Math 111-002
Assignment # 1 - Answers

The material in this assignment covers topics from Math 110.

1. Find the equation of the tangent line to the curve $y = x\sqrt{1+x^2}$ at the point $(1, \sqrt{2})$.

   Answer. The derivative is

$$y' = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}}.$$

   When $x = 1$, we get

$$y'(1) = \sqrt{2} + \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}.$$

   Thus the tangent line at $(1, \sqrt{2})$ is given by

$$y = \sqrt{2} + \frac{3}{\sqrt{2}}\,(x-1).$$

2. Let $h(x) = \sqrt{f(x)/g(x)}$. Find h' in terms of f, f', g, g'.

   Answer. We apply first the chain rule, and then the derivative of a quotient. We have

$$h'(x) = \frac{(f(x)/g(x))'}{2\sqrt{f(x)/g(x)}} = \frac{(f'(x)g(x) - f(x)g'(x)}{2\sqrt{f(x)/g(x)}g(x)^2} = \frac{(f'(x)g(x) - f(x)g'(x)}{2\sqrt{f(x)}\,g(x)^{3/2}}$$

3. Find $\int \sin x \, \cos(\cos x)\; dx$.

   Answer. We use substitution: take $u = \cos x$. Then $du = -\sin x\, dx$. So

$$\int \sin x \, \cos(\cos x)\; dx = -\int \cos u \, du = -\sin u = -\sin(\cos x) + C.$$

4. Find $\int_0^4 x\sqrt{16-3x}\, dx$.

Answer. The problem is that one cannot distribute the root over the sum. So we take $u = 16 - 3x$. Then $du = -3dx$; when $x = 0$, $u = 16$; and when $x = 4$, $u = 4$. Thus, using that $x = (16 - u)/3$,

$$\begin{aligned}
\int_0^4 x\sqrt{16-3x}\,dx &= -\frac{1}{3}\int_{16}^4 \frac{16-u}{3}u^{1/2}\,du = \frac{1}{9}\int_4^{16}(16u^{1/2} - u^{3/2})\,du \\
&= \frac{1}{9}\left(\frac{16u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}\right)\Bigg|_4^{16} = \frac{1}{9}\left(\frac{32u^{3/2}}{3} - \frac{2u^{5/2}}{5}\right)\Bigg|_4^{16} \\
&= \frac{1}{9}\left(\frac{32\times 16^{3/2}}{3} - \frac{2\times 16^{5/2}}{5} - \frac{32\times 4^{3/2}}{3} + \frac{2\times 4^{5/2}}{5}\right) \\
&= \frac{1}{9}\left(\frac{32\times 64}{3} - \frac{2\times 1024}{5} - \frac{32\times 8}{3} + \frac{2\times 32}{5}\right) \\
&= \frac{64}{9}\left(\frac{32}{3} - \frac{32}{5} - \frac{4}{3} + \frac{1}{5}\right) \\
&= \frac{64}{135}(160 - 96 - 20 + 3) = \frac{3008}{135}.
\end{aligned}$$

5. Calculate the area of the region enclosed by the two curves $y = 1 + \sqrt{x}$, $y = (3 + x)/3$.

   Answer. Intersection points: if $1 + \sqrt{x} = (3 + x)/3 = 1 + x/3$, we get $\sqrt{x} = x/3$. The solutions to this are $x = 0$ and $x = 9$. We can see from a plot or by checking on a point between 0 and 9 that $1 + \sqrt{x}$ is above the line. So the area is

$$\begin{aligned}
\int_0^9 [1 + \sqrt{x} - (1 + x/3)]\,dx &= \int_0^9 (\sqrt{x} - x/3)\,dx \\
&= \left(\frac{2x^{3/2}}{3} - \frac{x^2}{6}\right)\Bigg|_0^9 = \left(\frac{2\times 9^{3/2}}{3} - \frac{9^2}{6}\right) \\
&= \left(\frac{54}{3} - \frac{27}{2}\right) = 27\left(\frac{2}{3} - \frac{1}{2}\right) \\
&= \frac{27}{6} = \frac{9}{2}.
\end{aligned}$$

6. Use the Midpoint Rule, with $n = 4$, to approximate

$$\int_0^1 (x^2 + 1)\,dx.$$

Find the actual value of the integral and compare with the value you obtained.

Answer. With $n = 4$, the partition of the interval $[0, 1]$ will be given by

$$x_0 = 0, \quad x_1 = 0.25, \quad x_2 = 0.5, \quad x_3 = 0.75, \quad x_4 = 1.$$

So the midpoints will be

$$x_1' = 0.125, \quad x_2' = 0.375, \quad x_3' = 0.625, \quad x_4' = 0.875$$

The approximation is then

$$\begin{aligned}\sum_{j=1}^{4} f(x_j')\,\frac{1}{4} &= \frac{0.125^2 + 1 + 0.375^2 + 1 + 0.625^2 + 1 + 0.875^2 + 1}{4} \\ &= \frac{4 + 0.015625 + 0.140625 + 0.390625 + .7665625}{4} \\ &= \frac{5.3125}{4} = 1.328125.\end{aligned}$$

The actual value of the integral is

$$\int_0^1 (x^1 + 1)\,dx = \frac{1}{3} + 1 = \frac{4}{3}.$$

The error from the approximation is then

$$\frac{4}{3} - 1.328125 = 0.00508\bar{3},$$

so about half of one percent.