Math 111-002

Assignment # 2 - Answers

1. The region surrounded by the given curves is rotated around the specified axis. Find the volume of the resulting solid (you'll have to decide which method to use).

   (a) $y = \sqrt{x-1}$, $x = 2$, $x = 5$, $y = 0$, x-axis.
   Answer. We use disks. Since $x - 1 > 0$ on $[2,5]$,
   $$V = \pi \int_2^5 (\sqrt{x-1})^2\,dx = \pi \int_2^5 (x-1)\,dx = \pi \left(\frac{x^2}{2} - x\right)\Big|_2^5$$
   $$= \pi\left(\frac{25}{2} - 5 - \frac{4}{2} + 2\right) = \frac{15\pi}{2}.$$

   (b) $y = x^{2/3}$, $x = 1$, $y = 0$, y-axis.
   Answer. Using disks, we get $x = y^{3/2}$. Also, when $x = 1$ we get $y = 1$. So
   $$V = \pi \int_0^1 (1^2 - (y^{3/2})^2)\,dy = \pi \int_0^1 (1 - y^3)\,dy = \pi\left(1 - \frac{1}{4}\right) = \frac{3\pi}{4}.$$
   We can also do it using cylindrical shells. In this case,
   $$V = 2\pi \int_0^1 x\,x^{2/3}\,dx = 2\pi \int_0^1 x^{5/3}\,dx = 2\pi\,\frac{1}{8/3} = \frac{3\pi}{4}.$$

   (c) $y = x$, $y = 0$, $x = 2$, $x = 4$, y-axis.
   Using disks (washers, properly), we need to divide our region. For $0 \le y \le 2$, the curves bounding the region are $x = 4$ and $x = 2$. For $2 \le y \le 4$, the curves are $x = 4$ and $x = y$. The volume of the whole solid will be the sum of the volumes generated by rotating each of these two regions. So
   $$V = \pi \int_0^2 (4^2 - 2^2)\,dy + \pi \int_2^4 (4^2 - y^2)\,dy$$
   $$= 24\pi + \pi\left(16y - \frac{y^3}{3}\right)\Big|_2^4 = 24\pi + \pi\left(64 - \frac{64}{3} - 32 + \frac{8}{3}\right)$$
   $$= 24\pi + \pi\left(32 - \frac{56}{3}\right) = 24\pi + \frac{40\pi}{3} = \frac{112\pi}{3}.$$

Using cylindrical shells,

$$V = 2\pi \int_2^4 x\,x\,dx = 2\pi \int_2^4 x^2\,dx = 2\pi \left(\frac{x^3}{3}\right)\bigg|_2^4 = 2\pi \left(\frac{4^3}{3} - \frac{2^3}{3}\right) = \frac{112\pi}{3}.$$

(d) $y = (x+1)^2$, $y = 0$, $x = 0$, y-axis.
Using disks, when $x = 0$ we have $y = (0+1)^2 = 1$. Since $y > 0$, we get $x = \sqrt{y} - 1$, and y is $0 \le y \le 1$. Then

$$\begin{aligned} V &= \pi \int_0^1 (\sqrt{y} - 1)^2\,dy = \pi \int_0^1 (y - 2\sqrt{y} + 1)\,dy \\ &= \pi \left(\frac{y^2}{2} - \frac{4y^{3/2}}{3} + y\right)\bigg|_0^1 = \pi \left(\frac{3}{2} - \frac{4}{3}\right) = \frac{\pi}{6}. \end{aligned}$$

Using cylindrical shells, we need to put a minus sign in from of the integral. This comes from the fact that we are using negative x, and in the deduction of the formula we used x as the radius of a cylinder, so we were assuming it was positive.

$$\begin{aligned} V &= -2\pi \int_{-1}^0 x(x+1)^2\,dx = -2\pi \int_{-1}^0 (x^3 + 2x^2 + x)\,dx \\ &= -2\pi \left(\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2}\right)\bigg|_{-1}^0 = -2\pi \left(-\frac{1}{4} + \frac{2}{3} - \frac{1}{2}\right) = \frac{\pi}{6}. \end{aligned}$$

(e) $y = 4x - x^2$, $y = 8x - 2x^2$, y-axis.
Answer. The two curves intersect when $x = 0$ and $x = 4$. This would be hard to do by disks/washers, so we just use cylindrical shells.

$$\begin{aligned} V &= 2\pi \int_0^4 x(8x - 2x^2 - (4x - x^2))\,dx = 2\pi \int_0^4 (4x^2 - x^3)\,dx \\ &= 2\pi \left(\frac{4^4}{3} - \frac{4^4}{4}\right) = \frac{2\pi \times 4^4}{12} = \frac{2\pi \times 4^3}{3} = \frac{128\pi}{3}. \end{aligned}$$

(f) $y = x^2 - 3x + 2$, $y = 0$, y-axis.
Answer. By disks: the volume will not change if we do a symmetry around the x-axis: so we may consider $y = -x^2 + 3x - 2$,

which is now positive between its two roots 1 and 2. Solving for x, we will have two halves, namely

$$x = \frac{-3 \pm \sqrt{9 - 4(-2-y)(-1)}}{-2} = \frac{3 \pm \sqrt{1-4y}}{2}.$$

The highest point in the parabola occurs when $x = 3/2$ and is

$$-\left(\frac{3}{2}\right)^2 + \frac{9}{2} - 2 = -\frac{9}{4} + \frac{9}{2} - 2 = \frac{-9+18-8}{4} = \frac{1}{4}.$$

Now

$$\begin{aligned} V &= \pi \int_0^{1/4} \left[\left(\frac{3+\sqrt{1-4y}}{2} \right)^2 - \left(\frac{3-\sqrt{1-4y}}{2} \right)^2 \right] dy \\ &= 3\pi \int_0^{1/4} \sqrt{1-4y}\, dy = 3\pi \left(-\frac{(1-4y)^{3/2}}{6} \right) \Big|_0^{1/4} \\ &= \frac{3\pi}{6} = \frac{\pi}{2}. \end{aligned}$$

Things are a lot easier when doing cylindrical shells:

$$\begin{aligned} V &= 2\pi \int_1^2 x(-x^2+3x-2)\, dx = 2\pi \int_1^2 (-x^3+3x^2-2x)\, dx \\ &= 2\pi \left(-\frac{x^4}{4} + x^3 - x^2 \right) \Big|_1^2 = 2\pi \left(-\frac{16}{4} + 8 - 4 + \frac{1}{4} - 1 + 1 \right) \\ &= \frac{2\pi}{4} = \frac{\pi}{2}. \end{aligned}$$