Math 111-002

Assignment # 3

1. Determine whether each function is one-to-one:

   (a) $f(x) = 10 - 3x$;

   Answer. Since the graph of f is a line of nonzero slope, it is cut only once by every horizontal line. So f is one-to-one.

   One could also check that if $f(a) = f(b)$, then $a = b$. Indeed, if $10 - 3a = 10 - 3b$, then $3a = 3b$ and so $a = b$. So, no value of the function can be repeated: f is one-to-one.

   A third way is to notice that $f'(x) = -3$, so x is strictly decreasing and thus one-to-one.

   Answer. This function is not one-to-one, because for instance $f(1) = f(-1)$.

   (b) $f(x) = 4 + \cos x$, $0 \leq x \leq \pi$.

   Since f is differentiable, we can look at its derivative. We have $f'(x) = -\sin x$. On the interval $[0, \pi]$ the sin is positive, so $f'(x) \leq 0$ for every x in $[0, \pi]$. As f' is zero only at 0 and π, f is monotone, and so it is one-to-one.

2. If $f(x) = x + \cos x$, find $f^{-1}(1)$ and $(f^{-1})'(1)$.

   Answer. First, $f'(x) = 1 - \sin x$. The point where f takes the value 1 is 0 (because $f(0) = 0 + \cos 0 = 0 + 1 = 1$). That is, $f^{-1}(1) = 0$. Then

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{1 - \sin 0} = \frac{1}{1} = 1.$$

3. Find the inverse function of $f(x) = \frac{4x-1}{2x+5}$. Calculate the derivative of f^{-1} explicitly. Use the Inverse Function Theorem to calculate $(f^{-1})'(-1/5)$ and compare its value with the one from the derivative you found.

   Answer. First we find the inverse function. If $y = (4x - 1)/(2x + 5)$, then

$$\begin{aligned} y &= \frac{4x-1}{2x+5} \\ (2x+5)y &= 4x-1 \\ 2xy-4x &= -1-5y \\ x &= -\frac{1+5y}{2y-4}. \end{aligned}$$

So $f^{-1}(x) = -\frac{1+5x}{2x-4}$, and its derivative is

$$(f^{-1})'(x) = -\frac{22}{(2x-4)^2};$$

thus $(f^{-1})'(-1/5) = -\frac{22}{(-2/5-4)^2} = \frac{25}{22}$.

If now we use the Inverse Function Theorem: note that $-1/5 = f(0)$; we need f', which is $f'(x) = 22/(2x+5)^2$ so

$$(f^{-1})'(-1/5) = \frac{1}{f'(0)} = \frac{1}{22/5^2} = \frac{25}{22}.$$

4. Find $(f^{-1})'(a)$.

   (a) $f(x) = x^3 + 3\sin x + 2\cos x$, $a = 2$.
   Answer. $f(0) = 2$; $f'(x) = 3x^2 + 3\cos x - 2\sin x$. So

   $$(f^{-1})'(2) = \frac{1}{f'(0)} = \frac{1}{0+3-0} = \frac{1}{3}.$$

   (b) $f(x) = \sqrt{x^3 + x^2 + x + 1}$, $a = 2$.
   Answer. $2 = f(1)$, and $f'(x) = \frac{3x^2+2x+1}{2\sqrt{x^3+x^2+x+1}}$. So

   $$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{(3+2+1)/(2\sqrt{1+1+1+1})} = \frac{4}{6} = \frac{2}{3}.$$

Remark: *note that there is no obvious recipe to find the number $f^{-1}(a)$. Both of the previous questions are doable because a has been chosen in such a way that it is very clear which number b satisfies $f(b) = a$.*

5. Suppose h^{-1} is the inverse function of a differentiable function h. If $h(2) = 3$, and $h'(2) = -5$, find $(h^{-1})'(3)$.

   Answer. By the Inverse Function Theorem,

$$(h^{-1})'(3) = \frac{1}{h'(2)} = -\frac{1}{5}.$$

6. Differentiate the function:

   (a) $H(t) = \ln\sqrt{\dfrac{3t^2+2}{t-1}}$

   Answer. Note that $H(t) = \frac{1}{2}\ln\frac{3t^2+2}{t-1} = \frac{1}{2}(\ln(3t^2+2) - \ln(t-1)$. So

$$H'(t) = \frac{1}{2}\left(\frac{6t}{3t^2+2} - \frac{1}{t-1}\right)$$

   (b) $f(t) = \sin(\ln t)$

   Answer. We use the chain rule:

$$f'(t) = \frac{1}{t}\cos(\ln t)$$

   (c) $h(z) = \dfrac{1+\ln z}{1-\ln z}$

   Answer.

$$h'(z) = \frac{\frac{1}{z}(1-\ln z) - (1+\ln z)(-\frac{1}{z})}{(1-\ln z)^2} = \frac{2}{z(1-\ln z)^2}$$

   (d) $f(x) = \dfrac{(x^5+3x)^4\cos^2 x}{x^{1/5}}$

   Answer. This is good candidate for logarithmic differentiation:

$$\ln f(x) = 4\ln(x^5+3x) + 2\ln(\cos x) - \frac{1}{5}\ln x;$$

   so

$$(\ln f(x))' = \frac{20x^4+12}{x^5+3x} - \frac{2\sin x}{\cos x} - \frac{1}{5x}.$$

   Then

$$f'(x) = f(x)(\ln f(x))' = \frac{(x^5+3x)^4\cos^2 x}{x^{1/5}}\left(\frac{20x^4+12}{x^5+3x} - \frac{2\sin x}{\cos x} - \frac{1}{5x}\right)$$

(e) $f(x) = \sqrt[4]{\dfrac{x^3+1}{x^3-1}}$

Answer. Another good one for logarithmic differentiation:

$$\ln f(x) = \frac{1}{4}\left(\ln(x^3+1) - \ln(x^3-1)\right),$$

$$(\ln f(x))' = \frac{1}{4}\left(\frac{3x^2}{x^3+1} - \frac{3x^2}{x^3-1}\right),$$

so

$$f'(x) = f(x)(\ln f(x))' = \frac{1}{4}\sqrt[4]{\frac{x^3+1}{x^3-1}}\left(\frac{3x^2}{x^3+1} - \frac{3x^2}{x^3-1}\right) = \frac{-6x^2}{4(x^6-1)}\sqrt[4]{\frac{x^3+1}{x^3-1}}$$

7. Evaluate the integral:

(a) $\displaystyle\int_e^{e^2} \frac{2}{x}\,dx$

Answer.

$$\int_e^{e^2} \frac{2}{x}\,dx = \quad 2\ln x\Big|_e^{e^2} = 2\ln e^2 - 2\ln e = 4 - 2 = 2.$$

(b) $\displaystyle\int_e^6 \frac{dx}{x\ln x}$

Answer. We can do this one by substitution. Let $u = \ln x$. Then $du = dx/x$, so

$$\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{du}{u} = \quad \ln u\Big|_1^{\ln 6} = \ln(\ln 6)$$

(c) $\displaystyle\int \frac{x^2+1}{x^3+3x+1}\,dx$

Answer. By substitution: let $u = x^3+3x+1$; then $du = 3(x^2+1)dx$. So

$$\int \frac{x^2+1}{x^3+3x+1}\,dx = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln u = \frac{1}{3}\ln(x^3+3x+1) + C$$

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(d) $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$

Answer. This is very similar to the previous one: if we let $u = 1 + \sin x$, then $du = \cos x\,dx$. So

$$\int \frac{\cos x}{1+\sin x}\,dx = \int \frac{du}{u} = \ln u = \ln(1+\sin x) + C$$