Math 111-002
Assignment # 4 - Answers

1. Differentiate the function. Simplify where possible.

   (a) $f(t) = \dfrac{e^t}{t^2+1}$

   Answer. $f'(t) = \dfrac{e^t(t^2+1) - 2te^t}{(t^2+1)^2} = \dfrac{e^t(t-1)^2}{(t^2+1)^2}$

   (b) $f(x) = e^x \ln x$

   Answer. $f'(x) = e^x \ln x + \dfrac{e^x}{x} = e^x \left(\ln x + \dfrac{1}{x} \right)$.

   (c) $g(u) = \dfrac{e^u + e^{-u}}{e^u - e^{-u}}$

   Answer. $g'(u) = \dfrac{(e^u - e^{-u})^2 - (e^u + e^{-u})^2}{(e^u - e^{-u})^2} = 1 - g(u)^2$

   (d) $h(s) = \sqrt{1 + se^{-s}}$

   Answer.
$$h'(s) = \frac{e^{-s} - se^{-s}}{2\sqrt{1+se^s}} = \frac{e^{-s}(1-s)}{2\sqrt{1+se^s}}$$

   (e) $f(x) = \log_7(xe^x)$

   Answer. $f(x) = \ln(xe^x)/\ln 7 = (\ln x + \ln e^x)/\ln 7 = (\ln x + x)/\ln 7$. So $f'(x) = (1/x + 1)/\ln 7$

   (f) $h(t) = t^{\sin t}$

   Answer. $h(t) = e^{\sin t \, \ln t}$. So $h'(t) = (\cos t \, \ln t + \sin t/t)e^{\sin t \, \ln t} = (\cos t \, \ln t + \sin t/t)t^{\sin t}$

   (g) $g(s) = (\cos s)^{\ln s}$

   Answer. $g(s) = e^{\ln s \, \ln(\cos s)}$, so
$$g'(s) = \left(\frac{1}{s} \ln(\cos s) - (\ln s) \frac{\sin s}{\cos s} \right) e^{\ln s \, \ln(\cos s)} = \left(\frac{1}{s} \ln(\cos s) - (\ln s) \frac{\sin s}{\cos s} \right) (\cos s)^{l}$$

   (h) $f(t) = \arctan \sqrt{\dfrac{1-t}{1+t}}$

Answer.

$$\begin{aligned} f'(t) &= \frac{1}{1+\left(\sqrt{\frac{1-t}{1+t}}\right)^2} \frac{1}{2\sqrt{\frac{1-t}{1+t}}} \frac{-(1+t)-(1-t)}{(1+t)^2} \\ &= \frac{1}{1+\frac{1-t}{1+t}} \frac{1}{2\sqrt{\frac{1-t}{1+t}}} \frac{(-2)}{(1+t)^2} \\ &= \frac{(-1)}{2\sqrt{1+t}\sqrt{1-t}} = -\frac{1}{2\sqrt{1-t^2}} \end{aligned}$$

Comment: note that the last derivative $f'(t)$ is the same (save for the 1/2) as that of the inverse cosine function. In other words, there exists a number c such that $2\arctan\sqrt{\frac{1-t}{1+t}} = \arccos t + c$; putting $t = 0$, we get $0 = \frac{\pi}{2} + c$. So $c = -\pi/2$. We have proven that

$$\arctan\sqrt{\frac{1-t}{1+t}} = \frac{1}{2}\arccos t - \frac{\pi}{4}$$

(i) $f(t) = \arcsin\left(\dfrac{2+3\cos x}{3-2\cos x}\right)$.

Answer. Using the Chain Rule,

$$f'(t) = \frac{1}{\sqrt{1-\left(\dfrac{2+3\cos x}{3-2\cos x}\right)^2}} \frac{-3\sin x(3-2\cos x)-(2+3\cos x)2\sin x}{(3-2\cos x)^2}$$

$$= \frac{-13\sin x}{|3-2\cos x|\sqrt{|1-5\cos x|\,(5+\cos x)}}$$

2. Evaluate the integral:

(a) $\displaystyle\int \frac{(1+\sqrt{x})^4}{\sqrt{x}}\,dx$

Answer. This one is not really related to logarithms: $1/\sqrt{x}$ is "almost" the derivative of $\sqrt{x}$, so letting $u = 1+\sqrt{x}$, $du = (1/2\sqrt{x})\,dx$; then

$$\int \frac{(1+\sqrt{x})^4}{\sqrt{x}}\,dx = 2\int u^4\,du = \frac{2u^5}{5} + C = \frac{2(1+\sqrt{x})^5}{5} + C$$

(b) $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

Answer. After taking $u = e^x$, $du = e^x\,dx$, so

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \arctan u + C = \arctan e^x + C$$

(c) $\displaystyle\int_0^1 xe^{-x^2}\,dx$

Answer. Again by substitution, $u = -x^2$, then $du = -2x\,dx$. So

$$\int_0^1 xe^{-x^2}\,dx = -\frac{1}{2}\int_0^{-1} e^u\,du = \quad \frac{1}{2}e^u\Big|_{-1}^{0} = \frac{1}{2}(1-e^{-1})$$

(d) $\displaystyle\int (x^6 + 6^x)\,dx$

Answer. $\displaystyle\int (x^6 + 6^x)\,dx = \frac{x^7}{7} + \frac{6^x}{\ln 6} + C$

(e) $\displaystyle\int x\,2^{x^2}\,dx$

Answer. Letting $u = x^2$, $du = 2xdx$. So

$$\int x\,2^{x^2}\,dx = \frac{1}{2}\int 2^u\,du = \frac{2^u}{2\ln 2} + C = \frac{2^{x^2}}{2\ln 2} + C$$

(f) $\displaystyle\int \frac{3^x}{1+3^x}\,dx$

Answer. Letting $u = 1 + 3^x$, $du = (\ln 3)\,3^x\,dx$, so

$$\int \frac{3^x}{1+3^x}\,dx = \frac{1}{\ln 3}\int \frac{1}{u}\,du = \frac{\ln u}{\ln 3} + C = \frac{\ln(1+3^x)}{\ln 3} + C$$

(g) $\displaystyle\int \frac{3^x}{1+3^{2x}}\,dx$

Answer. Let $u = 3^x$, $du = (\ln 3)\,3^x\,dx$, so

$$\int \frac{3^x}{1+3^{2x}}\,dx = \frac{1}{\ln 3}\int \frac{1}{1+u^2}\,du = \frac{\arctan u}{\ln 3} + C = \frac{\arctan 3^x}{\ln 3} + C$$

(h) $\int \frac{x}{1+x^4}\,dx$

Answer. The key here is to recognize that $x^4 = (x^2)^2$. Putting $u = x^2$, $du = 2xdx$, and

$$\int \frac{x}{1+x^4}\,dx = \frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\,\arctan(u)+C = \frac{1}{2}\,\arctan(x^2)+C$$

(i) $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

Answer. If $u = e^x$, then $du = e^x dx$. So

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx = \int \frac{1}{\sqrt{1-u^2}}\,du = \arcsin(u)+C = \arcsin(e^x)+C$$

3. Simplify each expression:

(a) $\ln\sqrt{e}$

Answer. $\ln\sqrt{e} = \frac{1}{2}\,\ln e = \frac{1}{2}$.

(b) $e^{3\ln 2}$

Answer. $e^{3\ln 2} = e^{\ln 2^3} = 2^3 = 8$.

(c) $e^{x+\ln x}$

Answer. $e^{x+\ln x} = e^x\,e^{\ln x} = xe^x$.

(d) $\tan(\arcsin x)$.

Answer. We need to write the tangent in terms of the sine. From $\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$ we obtain $\tan^2 x = \frac{\sin^2 x}{1-\sin^2 x}$. Then

$$\tan^2(\arcsin x) = \frac{\sin^2 \arcsin x}{1-\sin^2 \arcsin x} = \frac{x^2}{1-x^2}.$$

We want to take the square root to solve for the tangent. Note that the domain of the arcsine is $[-1,1]$. At these endpoints the arcsine is respective $-\pi/2$ and $\pi/2$, where the tangent is not defined. So the domain is $(-1,1)$. When $x > 0$, we get from above that

$$\tan(\arcsin x) = \sqrt{\frac{x^2}{1-x^2}} = \frac{|x|}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}.$$

When $x < 0$, the tangent is negative, and this we lose when squaring. But $\tan(x) = -\tan(-x)$, and $\arcsin(-x) = -\arcsin(x)$ (from $\sin(-x) = -\sin x$). Then, for $x < 0$,

$$\begin{aligned}\tan(\arcsin x) &= -\tan(-\arcsin x)) = -\tan(\arcsin(-x)) \\ &= -\sqrt{\frac{(-x)^2}{1-(-x)^2}} = \frac{-|x|}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}.\end{aligned}$$

4. A sound so faint that it can just be heard as an intensity $I_0 = 10^{-12}$ watt/m^2. The loudness, in decibels (dB), of a sound with intensity I is then defined to be $L = 10\ \log_{10}(I/I_0)$. Amplified rock music is measured at 120dB, while the noise of a lawn mower at 106dB. Find the ratio of the intensity of the rock music to that of the mower.

   Answer. Solving from the equation, we get that $I = I_0\, 10^{L/10}$ If I_R and I_M are the intensities of the rock music and the mower respectively, we have

   $$120 = 10\ \log_{10}(I_R/I_0), \quad 106 = 10\ \log_{10}(I_M/I_0),$$

   and so $I_R = I_0\, 10^{120/10} = I_0\, 10^{12}$, $I_M = I_0\, 10^{106/10} = I_0\, 10^{10.6}$. Then

   $$\frac{I_R}{I_M} = \frac{I_0\, 10^{12}}{I_0\, 10^{10.6}} = \frac{10^{12}}{10^{10.6}} = 10^{12-10.6} = 10^{1.4} \simeq 25.12$$

5. The *secant*, *cosecant*, and *cotangent* functions are defined as

   $$\sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}, \quad \cot x = \frac{1}{\tan x}.$$

   Determine the domain and ranges of these three functions so that they admit inverses. Find the derivatives of these three inverses.

   Answer. The secant is defined for all real numbers with the exception of those where $\cos x = 0$, i.e. $(2n+1)\pi/2$, with n an integer. It has period 2π, and it will be increasing wherever the cosine is decreasing, and viceversa. So, on the interval $[0, \pi]$ the secant is increasing (and not defined at $x = \pi/2$). So we consider the domain $[0, \pi/2) \cup (\pi/2, \pi]$. Its range is $(-\infty, -1] \cup [1, \infty)$ (because the range of the cosine is $[-1, 1]$). So the inverse function $\operatorname{arcsec} x$ is defined on $(-\infty, -1] \cup [1, \infty)$. Let us find its derivative.

We have $\sec' x = -\frac{-\sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x}$, so if $y_0 = \sec x_0$ (and so $x_0 = \operatorname{arcsec} y_0$), then by the Inverse Function Theorem

$$\operatorname{arcsec}'(y_0) = \frac{1}{\sec' x_0} = \frac{1}{\frac{\sin x_0}{\cos^2 x_0}} = \frac{\cos^2 x_0}{\sin x_0}.$$

Now, $y_0 = \sec x_0 = 1/\cos x_0$, so $\cos x_0 = 1/y_0$. And then $\sin x_0 = \sqrt{1-\cos^2 x_0} = \sqrt{1-1/y_0^2}$. Then

$$\operatorname{arcsec}'(y_0) = \frac{\cos^2 x_0}{\sin x_0} = \frac{1}{y_0^2\sqrt{1-\frac{1}{y_0^2}}}$$

We have shown that

$$\operatorname{arcsec}'(x) = \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = \frac{1}{|x|\sqrt{x^2-1}}$$

(note that the domain we have chosen makes the expression inside the square root always positive!)

Now for the cosecant. The situation is very similar (as happened with the arcsin and arccos functions). So we could repeat a similar argument to the one we have just used. Only in this case the domain of the cosecant will be $[-\pi/2, 0) \cup (0, \pi/2)$ (because we cannot allow the sin to be zero). So, if $\operatorname{arccsc} x_0 = y_0$, then using that $\csc' x = -\frac{\cos x}{\sin^2 x}$ we get

$$\operatorname{arccsc}'(y_0) = \frac{1}{\csc' x_0} = -\frac{\sin^2 x_0}{\cos x_0} = -\frac{1}{y_0^2\sqrt{1-\frac{1}{y_0^2}}},$$

so

$$\operatorname{arccsc}'(x) = -\frac{1}{x^2\sqrt{1-\frac{1}{x^2}}} = -\frac{1}{|x|\sqrt{x^2-1}}$$

Note that we have the relation $\operatorname{arccsc}' x = -\operatorname{arcsec}' x$. *We can deduce (as we did before with the inverse sine and cosine functions) that* $\operatorname{arcsec} x + \operatorname{arccsc} x = c$ *for some number* c. *Note that* $\sec \pi/4 = \sqrt{2}$, $\csc \pi/4 = \sqrt{2}$. *So putting* $x = \sqrt{2}$ *we get* $c = \pi/4 + \pi/4 = \pi/2$. *We have shown that*

$$\operatorname{arccsc} x = \frac{\pi}{2} - \operatorname{arcsec} x.$$

Finally, the cotangent. Since the tangent is increasing on $(-\pi/2, \pi/2)$, the cotangent will be decreasing on that interval. Its range will be $(-\infty, +\infty)$, and so this will be the domain of its inverse (with zero removed).

We have $\cot' x = \left(\frac{\cos x}{\sin x}\right)' = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$. So, if $\cot x_0 = y_0$, we have that $\tan x_0 = 1/y_0$, and

$$\operatorname{arccot}' y_0 = \frac{1}{-\frac{1}{\sin^2 x_0}} = -\sin^2 x_0.$$

We need to express the sine squared in terms of the tangent. Since

$$\sin^2 x = \tan^2 x \, \cos^2 x = \tan^2 x \,(1 - \sin^2 x),$$

solving for $\sin^2 x$ we get that $\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x}$. So

$$\operatorname{arccot}' y_0 = -\sin^2 x_0 = -\frac{\tan^2 x_0}{1 + \tan^2 x_0} = -\frac{1}{y_0^2\,(1 + \frac{1}{y_0^2})} = -\frac{1}{1 + y_0^2}.$$

We have shown that

$$\operatorname{arccot}' x = -\frac{1}{1 + x^2}.$$

We can use this to deduce that $\arctan x + \operatorname{arccot} x = \pi/2$.