Math 111-002
Assignment # 5 - Answers

1. Find the limit

(a) $\lim_{x\to\pi/2} \frac{\sin x}{x}$

Answer. Both functions are continuous and nonzero at $\pi/2$, so

$$\lim_{x\to\pi/2} \frac{\sin x}{x} = \frac{\sin \pi/2}{\pi/2} = \frac{2}{\pi}.$$

(b) $\lim_{x\to 0} \frac{\sin x}{x}$

Answer. Both numerator and denominator are continuous at 0, and the quotient is of the form 0/0 at $x = 0$. So we apply L'Hopital's rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1.$$

As the limit of the quotient of the derivatives exists, so does our original limit.

(c) $\lim_{x\to\infty} \frac{\sin x}{x}$

Answer. Since $-1 \le \sin x \le 1$ for any x,

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x},$$

so

$$0 = \lim_{x\to\infty} \left(-\frac{1}{x}\right) \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0,$$

from where we conclude (by the "squeeze rule") that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

(d) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$

Answer. Both numerator and denominator are continuous at 0, and the quotient is of the form 0/0 at $x = 0$. So we apply L'Hopital's rule (three times):

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1 - \cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}.$$

(e) $\lim_{x\to 0} x - \sqrt{x^2 - x}$

Answer. We start by writing our expression as a quotient:

$$x - \sqrt{x^2 - x} = (x - \sqrt{x^2 - x})\frac{x + \sqrt{x^2 - x}}{x + \sqrt{x^2 - x}} = \frac{x^2 - (x^2 - x)}{x + \sqrt{x^2 - x}} = \frac{x}{x + \sqrt{x^2 - x}}.$$

Now this is a "0/0", and we can try with L'Hopital's rule:

$$\begin{aligned} \lim_{x\to 0} x - \sqrt{x^2 - x} &= \lim_{x\to 0} \frac{x}{x + \sqrt{x^2 - x}} = \lim_{x\to 0} \frac{1}{1 + \frac{2x-1}{2\sqrt{x^2-x}}} \\ &= \lim_{x\to 0} \frac{2\sqrt{x^2 - x}}{2\sqrt{x^2 - x} + 2x - 1} = 0 \end{aligned}$$

(note that the last limit is not indeterminate, because the denominator goes to -1).

(f) $\lim_{x\to 0} x \sin(1/x)$

Answer. Since the sine function is always bounded between -1 and 1, we have

$$-x \leq x \sin(1/x) \leq x,$$

so $\lim_{x\to 0} x \sin(1/x) = 0$ by the squeeze rule.

Note that if you try to apply L'Hopital here, you get

$$\lim_{x\to 0} x \sin(1/x) = \lim_{x\to 0} \frac{\sin(1/x)}{1/x} = \lim_{x\to 0} \frac{-\frac{1}{x^2}\cos(1/x)}{-\frac{1}{x^2}} = \lim_{x\to 0} \cos(1/x)$$

and this last limit does not exist (so it is not legit to apply L'Hopital!)

(g) $\lim_{x\to 0} \frac{x + \tan 2x}{x - \tan 2x}$

Answer. Both numerator and denominator are continuous at 0, and the quotient is of the form 0/0 at $x = 0$. So we apply L'Hopital's rule:

$$\lim_{x\to 0} \frac{x + \tan 2x}{x - \tan 2x} = \lim_{x\to 0} \frac{1 + \frac{2}{\cos^2 2x}}{1 - \frac{2}{cos^2 2x}} = \frac{1+2}{1-2} = -3.$$

(h) $\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

Answer. First we write the limit as quotient,

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\,\ln x}\right),$$

and then we use L'Hopital (twice; in both cases the limit is still of the form 0/0):

$$\lim_{x\to 1}\left(\frac{x-1-\ln x}{(x-1)\,\ln x}\right) = \lim_{x\to 1}\left(\frac{1-\frac{1}{x}}{\ln x\, + \frac{x-1}{x}}\right) = \lim_{x\to 1}\left(\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}\right) = \frac{1}{1+1} = \frac{1}{2}.$$

(i) $\lim_{x\to 0}\frac{e^{2x}-e^{3x}}{x}$

Answer. Both numerator and denominator are continuous at 0, and the quotient is of the form 0/0 at $x = 0$. So we apply L'Hopital's rule:

$$\lim_{x\to 0}\frac{e^{2x}-e^{3x}}{x} = \lim_{x\to 0}\frac{2e^{2x}-3e^{3x}}{1} = 2-3 = -1.$$

(j) $\lim_{x\to 0}\frac{2^x-1}{x}$

Answer. Both numerator and denominator are continuous at 0, and the quotient is of the form 0/0 at $x = 0$. So we apply L'Hopital's rule:

$$\lim_{x\to 0}\frac{2^x-1}{x} = \lim_{x\to 0}\frac{(\ln 2)\,2^x}{1} = \ln 2.$$