Math 111-002
Assignment # 6

Please remember that the assignment consists of only a sample of the kind of questions you are supposed to be able to do. It is **not** a safe practice to just do the assignment, and that is why there is a list of "suggested practice problems" in the course web page.

1. Evaluate the integral.

(a) $\displaystyle\int_0^1 (x^2+3)\, e^x\, dx$

Answer. By parts (twice),

$$\begin{aligned}\int_0^1 (x^2+3)\, e^x\, dx &= (x^2+3)\, e^x\Big|_0^1 - \int_0^1 2x\, e^x\, dx = (4e-3) - 2\left(xe^x\Big|_0^1 - \int_0^1 e^x\, dx\right)\\ &= 4e-3-2(e-(e-1)) = 4e-5.\end{aligned}$$

(b) $\displaystyle\int_1^2 \sqrt{u}\, \ln u\, du$

Answer. By parts,

$$\begin{aligned}\int_1^2 \sqrt{u}\, \ln u\, du &= \frac{2}{3}u^{3/2}\ln u\Big|_1^2 - \frac{2}{3}\int_1^2 u^{3/2}\frac{1}{u}\, du = \frac{4\sqrt{2}}{3}\ln 2 - \frac{2}{3}\int_1^2 \sqrt{u}\, du\\ &= \frac{4\sqrt{2}}{3}\ln 2 - \frac{2}{3}\frac{2}{3}u^{3/2}\Big|_1^2 = \frac{4\sqrt{2}}{3}\ln 2 - \frac{4}{9}(2\sqrt{2}-1).\end{aligned}$$

(c) $\displaystyle\int_0^1 x\, \arctan(x)\, dx$

Answer. By Parts,

$$\begin{aligned}\int_0^1 x\, \arctan(x)\, dx &= \frac{x^2}{2}\arctan x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}\, dx = \frac{\pi}{8} - \frac{1}{2}\int_0^1 \left(1-\frac{1}{1+x^2}\right) dx\\ &= \frac{\pi}{8} - \frac{1}{2}(x-\arctan x)\Big|_0^1 = \frac{\pi}{8} - \frac{1}{2}\left(1-\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

(d) $\displaystyle\int e^{\sqrt{x}}\, dx$

Answer. Doing parts at once is not useful in this case, but after a substitution the integral can be done easily: let $u=\sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\, dx$, and after the substitution we do parts:

$$\int e^{\sqrt{x}}\, dx = \int 2ue^u\, du = 2ue^u - 2\int e^u\, du = 2ue^u - 2e^u = 2e^{\sqrt{x}}(\sqrt{x}-1)$$

(e) $\displaystyle\int_0^1 \frac{t^3}{\sqrt{2+t^2}}\,dt$

Answer. We start with the substitution $u = 2 + t^2$, and we get

$$\begin{aligned}\int_0^1 \frac{t^3}{\sqrt{2+t^2}}\,dt &= \frac{1}{2}\int_2^3 \frac{u-2}{\sqrt{u}}\,du = \frac{1}{2}\int_2^3 (u^{1/2} - 2u^{-1/2})\,du \\ &= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 4u^{1/2}\right)\Bigg|_2^3 = -\sqrt{3} + \frac{4}{3}\sqrt{2}.\end{aligned}$$

It is also possible to do this integral by trigonometric substitution (and more work!): if we let $t = \sqrt{2}\tan v$, then $dt = \sqrt{2}\sec^2 v\,dv$, and

$$\begin{aligned}\int_0^1 \frac{t^3}{\sqrt{2+t^2}}\,dt &= \int_{t=0}^{t=1} \frac{2\tan^3 v}{\sec v}\sqrt{2}\sec^2 v\,dv = 2\sqrt{2}\int_{t=0}^{t=1} \tan^3 v\,\sec v\,dv \\ &\quad \text{(now we substitute } w = \sec v \text{, with } dw = \tan v\,\sec v\,dv\text{)} \\ &= 2\sqrt{2}\int_{t=0}^{t=1}(w^2-1)dw = 2\sqrt{2}\left(\frac{w^3}{3} - w\right)\Bigg|_{t=0}^{t=1} \\ &= 2\sqrt{2}\left(\frac{\sec^3(\arctan\frac{t}{\sqrt{2}})}{3} - \sec(\arctan\frac{t}{\sqrt{2}})\right)\Bigg|_0^1 \\ &\quad \text{(now you need some trigonometry to determine that} \\ &\quad \sec(\arctan\tfrac{1}{\sqrt{2}}) = \tfrac{\sqrt{3}}{\sqrt{2}}\text{: this comes from the square triangle} \\ &\quad \text{with sides } 1, \sqrt{2}, \sqrt{3}\text{)} \\ &= 2\sqrt{2}\left(\frac{(\sqrt{3})^3}{3(\sqrt{2})^3} - \frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{3} + 1\right) = \sqrt{3} - 2\sqrt{3} + \frac{4}{3}\sqrt{2} \\ &= -\sqrt{3} + \frac{4}{3}\sqrt{2}.\end{aligned}$$

Yet a third way is to do the integral by parts: indeed, if $f(t) = t^2$, $g(t) = \sqrt{2+t^2}$, then

$$\begin{aligned}\int_0^1 \frac{t^3}{\sqrt{2+t^2}}\,dt &= \int_0^1 f(t)\,g'(t)\,dt = t^2\sqrt{2+t^2}\Big|_0^1 - \int_0^1 2t\sqrt{2+t^2}\,dt \\ &= \sqrt{3} - \frac{2}{3}(2+t^2)^{3/2}\Big|_0^1 = \sqrt{3} - \frac{2}{3}3^{3/2} + \frac{2}{3}2^{3/2} \\ &= \sqrt{3} - 2\sqrt{3} - \frac{4}{3}\sqrt{2} = -\sqrt{3} + \frac{4}{3}\sqrt{2}.\end{aligned}$$

(f) $\displaystyle\int_0^{\pi/2} \cos^5 t\,dt$

Answer. If we substitute $u = \sin t$, then $du = \cos t\,dt$, and

$$\begin{aligned}\int_0^{\pi/2} \cos^5 t\,dt &= \int_0^1 (1-u^2)^2\,du = \int_0^1 (1 - 2u^2 + u^4)\,du = \left(u - \frac{2u^3}{3} + \frac{u^5}{5}\right)\Bigg|_0^1 \\ &= 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}.\end{aligned}$$

(g) $\displaystyle\int \tan^4 x \sec^4 x\, dx$

Answer. If we let $u = \tan x$, then $du = \sec^2 x\, dx$, and using that $\sec^2 x = 1 + u^2$, we get

$$\int \tan^4 x \sec^4 x\, dx = \int u^4(1+u^2)\, du = \int (u^4+u^6)\, du = \frac{u^5}{5} + \frac{u^7}{7} = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7}.$$

(h) $\displaystyle\int \frac{1}{1-\cos t}\, dt$

Answer. First we multiply the integrand, top and bottom, by $1 + \cos t$, to get

$$\int \frac{1}{1-\cos t}\, dt = \int \frac{1+\cos t}{1-\cos^2 t}\, dt = \int \frac{1+\cos t}{\sin^2 t}\, dt = \int \frac{1}{\sin^2 t}\, dt + \int \frac{\cos t}{\sin^2 t}\, dt.$$

Now we do the first integral in the last sum. We divide numerator and denominator by $\cos^2 t$, and we have, using $u = \tan t$, $du = \sec^2 t\, dt$,

$$\int \frac{1}{\sin^2 t}\, dt = \int \tan^{-2} t \sec^2 t\, dt = \int u^{-2}\, du = -u^{-1} = -\frac{1}{\tan t}.$$

For the second integral in the sum we use the substitution $w = \sin t$, $dw = \cos t\, dt$, so

$$\int \frac{\cos t}{\sin^2 t}\, dt = \int w^{-2}\, dw = -w^{-1} = -\frac{1}{\sin t}.$$

Putting the information together, we have

$$\int \frac{1}{1-\cos t}\, dt = -\frac{1}{\tan t} - \frac{1}{\sin t} = -\frac{1+\cos t}{\sin t}.$$

Another way of starting the integral is to use some trigonometry: since $\cos t = 2\cos^2 \frac{t}{2} - 1$, $1 - \cos t = 2(1 - \cos^2 \frac{t}{2}) = 2\sin^2 t/2$; then, with $u = t/2$, and using an integral we already calculated above,

$$\int \frac{1}{1-\cos t}\, dt = \frac{1}{2} \int \frac{1}{\sin^2 \frac{t}{2}}\, dt = \int \frac{1}{\sin^2 u}\, du = -\frac{1}{\tan u} = -\frac{1}{\tan \frac{t}{2}}.$$

It is not difficult to check that both answers are the same. Indeed, since $\sin t = 2\sin \frac{t}{2} \cos \frac{t}{2}$ and $\cos t = 2\cos^2 \frac{t}{2} - 1$, we have

$$\tan \frac{t}{2} = \frac{\sin t/2}{\cos t/2} = \frac{\sin t}{2\cos^2 t/2} = \frac{\sin t}{1+\cos t}.$$

Yet a third way to do it is to use a trick that works in many cases: every time we have a rational function on sin and cos we can try the substitution $v = \tan(t/2)$. Then $dv = \frac{1}{2} \sec^2 \frac{t}{2}$. So

$$\begin{aligned}\int \frac{dt}{1-\cos t} &= \int \frac{1}{2\sin^2 \frac{t}{2}} \cdot \frac{1}{\frac{1}{2}\sec^2 \frac{t}{2}}\, dv = \int \frac{\cos^2 \frac{t}{2}}{\sin^2 \frac{t}{2}}\, dv \\ &= \int \frac{\frac{1}{1+v^2}}{\frac{v^2}{1+v^2}}\, dv = \int \frac{1}{v^2}\, dv = -\frac{1}{v} = -\frac{1}{\tan \frac{t}{2}}\end{aligned}$$

(i) $\displaystyle\int_0^1 \sqrt{x^2+1}\,dx$

Answer. We put $x = \tan t$, $dx = \sec^2 t\,dt$, since $1+\tan^2 t = \sec^2 t$ and the secant is positive on the interval $[0, \pi/4]$, we get

$$\begin{aligned}\int_0^1 \sqrt{x^2+1}\,dx &= \int_0^{\pi/4} \sec^3 t\,dt = \frac{1}{2}\left(\sec t\,\tan t + \ln|\sec t + \tan t|\right)\Big|_0^{\pi/4}\\ &= \frac{1}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right)\end{aligned}$$

(the integral of the secant cube is done in section 7.2 in Stewart (example 8); it is done by parts and it requires the integral of the secant, which is done in the same page and also in class, Feb 14th).

(j) $\displaystyle\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx$

Answer. First we complete the square: $5-4x-x^2 = -(x^2+4x-5) = -(x^2+4x+4-9) = -((x+2)^2-9) = 9-(x+2)^2$. So, with $u = x+2$, $du = dx$,

$$\begin{aligned}\int \frac{1}{(5-4x-x^2)^{5/2}}\,dx &= \int \frac{1}{(9-(x+2)^2)^{5/2}}\,dx = \int \frac{1}{(9-u^2)^{5/2}}\,du\\ &\quad\text{(using the substitution } u = 3\sin t,\ du = 3\cos t)\\ &= \int \frac{3\cos t}{(9(1-\sin^2 t))^{5/2}}\,dt = \frac{1}{3^4}\int \frac{\cos t}{\cos^5 t}\,dt\\ &\quad\text{(using } v = \tan t,\ dv = \sec^2 t\,dt,\ \sec^2 t = 1+v^2)\\ &= \frac{1}{81}\int \sec^4 t\,dt = \frac{1}{81}\int (1+v^2)\,dv\\ &= \frac{1}{81}\left(v + \frac{v^3}{3}\right) = \frac{1}{81}\left(\tan t + \frac{\tan^3 t}{3}\right)\\ &\quad\text{(using that } \tan t = \tfrac{\sin t}{\sqrt{1-\sin^2 t}},\ \sin t = (x+2)/3)\\ &= \frac{1}{81}\left(\frac{x+2}{3\sqrt{1-\left(\frac{x+2}{3}\right)^2}} + \frac{(x+2)^3}{3^4\left(1-\left(\frac{x+2}{3}\right)^2\right)^{3/2}}\right)\\ &= \frac{1}{243}\left(\frac{3(x+2)}{\sqrt{9-(x+2)^2}} + \frac{(x+2)^3}{(9-(x+2)^2)^{3/2}}\right)\\ &= \frac{(x+2)(19-8x-2x^2)}{243(5-4x-x^2)^{3/2}}\end{aligned}$$

(k) $\displaystyle\int_0^2 x\sqrt{16-x^2}\,dx$

Answer. The easiest way is to do this by substitution, with $u = 16-x^2$, $du = -2xdx$:

$$\int_0^2 x\sqrt{16-x^2}\,dx = -\frac{1}{2}\int_{16}^{12} \sqrt{u}\,du = \frac{1}{3}u^{3/2}\Big|_{12}^{16} = \frac{1}{3}(64-24\sqrt{3}).$$

It is also possible to use trigonometric substitution: if $x = 4\sin t$, then $dx = 4\cos t\, dt$, and t varies between 0 and $\pi/6$.

$$\begin{aligned}
\int_0^2 x\sqrt{16-x^2}\,dx &= \int_0^{\pi/6} 4\sin t\, 4\cos t\, 4\cos t\, dt = 64\int_0^{\pi/6} \sin t\, \cos^2 t\, dt \\
&= -64\,\frac{\cos^3 t}{3}\bigg|_0^{\pi/6} = -64\left(\left(\frac{1}{3}\,\frac{\sqrt{3}}{2}\right)^3 - \frac{1}{3}\right) = \frac{1}{3}\,(64 - 24\sqrt{3}).
\end{aligned}$$

(l) $\displaystyle\int_0^2 x^2\sqrt{16-x^2}\,dx$

Answer. Again we do a trigonometric substitution: $x = 4\sin t$, $dx = 4\cos t\, dt$, and t varies between 0 and $\pi/6$ (because $\sin\pi/6 = 1/2$).

$$\begin{aligned}
\int_0^2 x^2\sqrt{16-x^2}\,dx &= \int_0^{\pi/6} 16\sin^2 t\, 4\cos t\, 4\cos t\, dt = 256\int_0^{\pi/6} \sin^2 t\, \cos^2 t\, dt \\
&\quad \text{(next we use that } \sin^2 t = \tfrac{1}{2}(1-\cos 2t),\ \cos^2 t = \tfrac{1}{2}(1+\cos 2t)\text{)} \\
&= 64\int_0^{\pi/6} (1-\cos 2t)(1+\cos 2t)\,dt = 64\int_0^{\pi/6} (1-\cos^2 2t)\,dt \\
&= 64\int_0^{\pi/6} \sin^2 2t\,dt = 32\int_0^{\pi/3} \sin^2 v\,dv \\
&= 32\,\frac{1}{2}\,(v - \sin v\ \cos v)\ \bigg|_0^{\pi/3} = 16\left(\frac{\pi}{3} - \frac{1}{2}\,\frac{\sqrt{3}}{2}\right) \\
&= 16\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)
\end{aligned}$$