Math 111-002
Assignment # 7

Please remember that the assignment consists of only a sample of the kind of questions you are supposed to be able to do. It is **not** a safe practice to just do the assignment, and that is why there is a list of "suggested practice problems" in the course web page.

1. Evaluate the integral

   (a) $\displaystyle\int \frac{x}{x-3}\,dx$

   Answer. Because the degree of the numerator is not smaller than that of the denominator, we first have to do long division. But in this case this is very simple, because $x = (x-3)+3$, so

   $$\frac{x}{x-3} = 1 + \frac{3}{x-3},$$

   and

   $$\int \frac{x}{x-3}\,dx = \int 1\,dx + \int \frac{3}{x-3}\,dx = x + 3\ln|x-3|$$

   (b) $\displaystyle\int_0^1 \frac{x-1}{x^2+3x+2}\,dx$

   Answer. First we find the roots of x^2+3x+2; these are

   $$\frac{-3+\sqrt{9-4\times 2}}{2} = -1, \quad \frac{-3-\sqrt{9-4\times 2}}{2} = -2.$$

   So $x^2+3x+2 = (x+1)(x+2)$. We setup partial fractions:

   $$\frac{x-1}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{(A+B)x + 2A + B}{(x+1)(x+2)}.$$

We deduce that $A + B = 1$, $2A + B = -1$. Subtracting the first equality from the second, we get that $A = -2$; the the first equality gives $B = 3$. Then

$$\begin{aligned}\int_0^1 \frac{x-1}{x^2+3x+2}\,dx &= -2\int_0^1 \frac{1}{x+1}\,dx + 3\int_0^1 \frac{1}{x+2}\,dx \\ &= -2\ln|x+1| + 3\ln|x+2|\,\Big|_0^1 \\ &= -2\ln 2 + 3\ln 3 - 3\ln 2 = -5\ln 2 + 3\ln 3.\end{aligned}$$

Another (longer) way: Although it is longer, the advantage of this approach is that you can always do it (look at the next question). First we complete the square:

$$x^2+3x+2 = x^2+2\,\frac{3}{2}\,x+\frac{9}{4}-\frac{9}{4}+2 = \left(x+\frac{3}{2}\right)^2-\frac{1}{4} = \frac{(2x+3)^2-1}{4}$$

Then

$$\begin{aligned}\int_0^1 \frac{x-1}{x^2+3x+2}\,dx &= 4\int_0^1 \frac{x-1}{(2x+3)^2-1}\,dx \\ &\quad \text{(substituting } u = 2x+3,\ du = 2dx\text{)} \\ &= 2\int_3^5 \frac{\frac{u-3}{2}-1}{u^2-1}\,du = \int_3^5 \frac{u-5}{u^2-1}\,du \\ &= \int_3^5 \frac{u}{u^2-1}\,du - 5\int_3^5 \frac{1}{u^2-1}\,du \\ &= I_1 - 5I_2.\end{aligned}$$

For I_1 we substitute $v = u^2 - 1$, $dv = 2udu$, and then

$$I_1 = \frac{1}{2}\int_8^{24} \frac{1}{v}\,dv = \frac{1}{2}\ln v\,\Big|_8^{24} = \frac{\ln 24 - \ln 8}{2} = \frac{1}{2}\ln\frac{24}{8} = \frac{1}{2}\ln 3.$$

For I_2, we have to do partial fractions (we did this integral twice in class!). Because it is a difference of squares, $u^2-1 = (u-1)(u+1)$. We set up the partial fractions as

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1} = \frac{(A+B)u + A - B}{u^2-1}.$$

We deduce that $A+B=0$ (so $B=-A$) and $A-B=1$. From this last equation we get $2A=1$, that is $A=1/2$, and so $B=-1/2$. Then

$$\begin{aligned} I_2 &= \int_3^5 \frac{1}{u^2-1}\,du = \frac{1}{2}\int_3^5 \frac{1}{u-1}\,du - \frac{1}{2}\int_3^5 \frac{1}{u+1}\,du \\ &= \frac{1}{2}\ln|u-1|\,\Big|_3^5 - \frac{1}{2}\ln|u+1|\,\Big|_3^5 = \frac{1}{2}(\ln 4-\ln 2) - \frac{1}{2}(\ln 6-\ln 4) \\ &= \frac{1}{2}(2\ln 2-\ln 2) - \frac{1}{2}(\ln 6-2\ln 2) = \ln 2 - \frac{1}{2}\ln 3 \end{aligned}$$

Finally,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = I_1-5I_2 = \frac{1}{2}\ln 3 - 5\ln 2 + \frac{5}{2}\ln 3 = 3\ln 3 - 5\ln 2$$

(c) $\displaystyle\int_0^1 \frac{x-1}{x^2+3x+3}\,dx$

Answer. The technique we used in the first solution of the previous question will not work because the polynomial has no real roots. So first we complete the square:

$$x^2+3x+3 = x^2+2\,\frac{3}{2}\,x+\frac{9}{4}-\frac{9}{4}+3 = \left(x+\frac{3}{2}\right)^2+\frac{3}{4} = \frac{3}{4}\left(\left(\frac{x+\frac{3}{2}}{\sqrt{3}/2}\right)^2+1\right)$$

(at the end we factored 3/4 out because we aim to get u^2+1).

Then

$$\begin{aligned}
\int_0^1 \frac{x-1}{x^2+3x+3}\,dx &= \frac{4}{3}\int_0^1 \frac{x-1}{\left(\frac{x+\frac{3}{2}}{\sqrt{3}/2}\right)^2+1}\,dx \\
&\quad \text{(substituting } u = 2(x+3/2)/\sqrt{3},\ du = 2/\sqrt{3}\,dx\text{)} \\
&= \frac{2}{\sqrt{3}}\int_{\sqrt{3}}^{5/\sqrt{3}} \frac{\frac{\sqrt{3}}{2}u-\frac{3}{2}-1}{u^2+1}\,du = \frac{2}{\sqrt{3}}\int_{\sqrt{3}}^{5/\sqrt{3}} \frac{\frac{\sqrt{3}}{2}u-\frac{5}{2}}{u^2+1}\,du \\
&= \int_{\sqrt{3}}^{5/\sqrt{3}} \frac{u}{u^2+1}\,du - \frac{5}{\sqrt{3}}\int_{\sqrt{3}}^{5/\sqrt{3}} \frac{1}{u^2+1}\,du \\
&= \frac{1}{2}\ln(u^2+1)\,\Big|_{\sqrt{3}}^{5/\sqrt{3}} - \frac{5}{\sqrt{3}}\arctan u\,\Big|_{\sqrt{3}}^{5/\sqrt{3}} \\
&= \frac{1}{2}\left(\ln\frac{28}{3}-\ln 4\right) - \frac{5}{\sqrt{3}}\left(\arctan\frac{5}{\sqrt{3}}-\arctan\sqrt{3}\right) \\
&= \frac{1}{2}\ln\frac{7}{3} - \frac{5}{\sqrt{3}}\left(\arctan\frac{5}{\sqrt{3}}-\arctan\sqrt{3}\right)
\end{aligned}$$

(d) $\displaystyle\int \frac{1}{x^3-1}\,dx$

Answer. Since the denominator has degree 3, we need to start by writing it as a product of polynomials of lesser degree. It is easy to see that 1 is a root of x^3-1. So we perfom long division to obtain $x^3-1=(x-1)(x^2+x+1)$. Since x^2+x+1 has no real roots, we cannot expect to reduce any more. Then we set up partial fractions:

$$\frac{1}{x^3-1} = \frac{A}{x-1}+\frac{Bx+C}{x^2+x+1} = \frac{(A+B)x^2+(A-B+C)x+A-C}{(x-1)(x^2+x+1)}.$$

Since the coefficient for x^2 has to be zero, we deduce that $A+B=0$, so $B=-A$. The coefficient for x has to zero too, so $0=A-B+C=2A+C$. And the independent coefficient has to be 1, so $A-C=1$, or $C=A-1$. Putting this in the second equation we get $0=3A-1$, so $A=1/3$, $B=-1/3$,

$C = 1/3 - 1 = -2/3$. So we have

$$\begin{aligned}\int \frac{1}{x^3-1}\,dx &= \frac{1}{3}\int \frac{1}{x-1}\,dx + \int \frac{-\frac{1}{3}\,x - \frac{2}{3}}{x^2+x+1}\,dx\\ &= \frac{1}{3}\int \frac{1}{x-1}\,dx - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx\\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\,I_2.\end{aligned}$$

To calculate I_2 we first complete the square, to get

$$\begin{aligned}x^2+x+1 &= x^2+x+\frac{1}{4}-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4}\left(\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)^2+1\right)\\ &= \frac{3}{4}\left(\left(\frac{2x+1}{\sqrt{3}}\right)^2+1\right).\end{aligned}$$

Now, letting $u = \frac{2x+1}{\sqrt{3}}$, we get $du = \frac{2}{\sqrt{3}}dx$ and

$$\begin{aligned}I_2 &= \frac{\sqrt{3}}{2}\int \frac{\frac{\sqrt{3}u-1}{2}+2}{\frac{3}{4}(u^2+1)}\,du = \frac{2}{\sqrt{3}}\int \frac{\frac{\sqrt{3}}{2}u+\frac{3}{2}}{u^2+1}\,du = \int \frac{u+\sqrt{3}}{u^2+1}\,du\\ &= \int \frac{u}{u^2+1}\,du + \int \frac{\sqrt{3}}{u^2+1}\,du = \frac{1}{2}\ln(u^2+1) + \sqrt{3}\arctan u\\ &\quad \text{(note that } u^2+1 = \tfrac{4}{3}(x^2+x+1))\\ &= \frac{1}{2}\ln\left(\frac{4}{3}(x^2+x+1)\right) + \sqrt{3}\arctan\frac{2x+1}{\sqrt{3}}\\ &= \ln\frac{2}{\sqrt{3}} + \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\arctan\frac{2x+1}{\sqrt{3}}.\end{aligned}$$

Because we are calculating antiderivatives and not integrals, we are not concerned about constants, and so we will drop the first (constant) term: we will still have an antiderivative.

Collecting the information we get

$$\begin{aligned}\int \frac{1}{x^3-1}\,dx &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\,I_2\\ &= \frac{1}{3}\ln|x-1| - \frac{1}{3}\left(\frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\arctan\frac{2x+1}{\sqrt{3}}\right)\\ &= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}.\end{aligned}$$

(e) $\displaystyle\int \frac{x^2+x+1}{(x^2+1)^2}\,dx$

Answer. The fraction is already reduced (the denominator is expressed in terms of a polynomial of degree two, and the numerator has less degree than the denominator). It's useful that the expression in the denominator appears in the numerator, so we can cancel:

$$\begin{aligned}\int \frac{x^2+x+1}{(x^2+1)^2}\,dx &= \int \frac{x^2+1+x}{(x^2+1)^2}\,dx = \int \left(\frac{1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx \\ &= \arctan x + \int \frac{x}{(x^2+1)^2}\,dx.\end{aligned}$$

And the last integral is suitable for substitution: if $u = x^2+1$, $du = 2xdx$, and so

$$\int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du = -\frac{1}{2u} = -\frac{1}{2(x^2+1)}.$$

Collecting results, we have that

$$\int \frac{x^2+x+1}{(x^2+1)^2}\,dx = \arctan x - \frac{1}{2(x^2+1)}.$$

(f) $\displaystyle\int \frac{1}{x(x^2+9)^2}\,dx$

Answer. We setup partial fractions:

$$\begin{aligned}\frac{1}{x(x^2+9)^2} &= \frac{A}{x}+\frac{Bx+C}{x^2+9}+\frac{Dx+E}{(x^2+9)^2} \\ &= \frac{(A+B)x^4+Cx^3+(18A+9B+D)x^2+(9C+E)x+81A}{x(x^2+9)^2}.\end{aligned}$$

We immediately conclude that $A=\frac{1}{81}$, $B=-A=-\frac{1}{81}$, $C=0$. From $9C+E=0$ we deduce that $E=0$. From the quadratic term, $D=-18A-9B=-\frac{18}{81}+\frac{9}{81}=-\frac{2}{9}+\frac{1}{9}=-\frac{1}{9}$. So

$$\frac{1}{x(x^2+9)^2}=\frac{1}{81x}-\frac{x}{81(x^2+9)}-\frac{x}{9(x^2+9)^2}$$

and now the integrals are easy (and similar to those in previous questions): we substitute $u = x^2 + 9$ in the last two, to get

$$\begin{aligned}
\int \frac{1}{x(x^2+9)^2}\,dx &= \int \left(\frac{1}{81x} - \frac{x}{81(x^2+9)} - \frac{x}{9(x^2+9)^2}\right) dx \\
&= \frac{1}{81}\ln x - \frac{1}{162}\ln(x^2+9) + \frac{1}{18(x^2+9)}
\end{aligned}$$

(g) $\displaystyle\int \frac{x^2+3x-1}{x^3+7x}\,dx$

Answer. The key observation here is that $x^3 + 7x = x(x^2+7)$. After that, it's all about setting up partial fractions:

$$\frac{x^2+3x-1}{x^3+7x} = \frac{A}{x} + \frac{Bx+C}{x^2+7} = \frac{(A+B)x^2 + Cx + 7A}{x^2+7}.$$

Comparing the coefficients for each power of x, we get that $A+B = 1$, $C = 3$, $7A = -1$. So $A = -\frac{1}{7}$, $B = 1 - A = 1 + \frac{1}{7} = \frac{8}{7}$. Then

$$\begin{aligned}
\int \frac{x^2+3x-1}{x^3+7x}\,dx &= \int \left(-\frac{1}{7x} + \frac{\frac{8x}{7}+3}{x^2+7}\right) dx = \int \left(-\frac{1}{7x} + \frac{8x+21}{7(x^2+7)}\right) dx \\
&= -\frac{1}{7}\int \frac{1}{x}\,dx + \frac{8}{7}\int \frac{x}{x^2+7}\,dx + 3\int \frac{1}{x^2+7}\,dx \\
&\quad \text{(we use the substitution } u = x^2+7 \text{ in the second integral)} \\
&= -\frac{1}{7}\ln x + \frac{4}{7}\ln(x^2+7) + \frac{3}{7}\int \frac{1}{\left(\frac{x}{\sqrt{7}}\right)^2 + 1}\,dx \\
&= -\frac{1}{7}\ln x + \frac{4}{7}\ln(x^2+7) + \frac{3}{\sqrt{7}}\arctan\left(\frac{x}{\sqrt{7}}\right).
\end{aligned}$$