Math 111-002
Assignment # 8 - Answers

1. Decide whether each integral is convergent or divergent. Evaluate those that are convergent.

(a) $\displaystyle\int_3^\infty \frac{x}{(x^2+2)^2}\,dx$

Answer. The integrand is always continuous, so the integral is only improper at infinity. Notice that the antiderivative can be easily found by using the substitution $u = x^2+2$. Then

$$\begin{aligned}\int_3^\infty \frac{x}{(x^2+2)^2}\,dx &= \lim_{N\to\infty}\int_3^N \frac{x}{(x^2+2)^2}\,dx = \lim_{N\to\infty} -\frac{1}{2(x^2+2)}\Bigg|_3^N \\ &= \lim_{N\to\infty} -\frac{1}{2(N^2+2)} + \frac{1}{2(3^2+2)} = \frac{1}{22}.\end{aligned}$$

(b) $\displaystyle\int_2^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

Answer. Once again the integrand is always continuous, so the integral is improper only at infinity. In this case, the substitution $u = \sqrt{x}$ shows that the antiderivative is $-2\,e^{-\sqrt{x}}$. Then

$$\int_2^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = -2\,e^{-\sqrt{x}}\Bigg|_2^\infty = 2\,e^{-\sqrt{2}} - 2\lim_{N\to\infty} e^{-\sqrt{N}} = 2\,e^{-\sqrt{2}}.$$

(c) $\displaystyle\int_{-\infty}^\infty x^3 e^{-x^4}\,dx$

Answer. Since the integral is improper on both limits, we have to split it in two, for example like $\displaystyle\int_{-\infty}^\infty x^3e^{-x^4}\,dx = \int_{-\infty}^0 x^3e^{-x^4}\,dx + \int_0^\infty x^3e^{-x^4}\,dx$. Noting that $x^3\,dx$ is basically the differential of x^4, we go with the substitution $u = x^4$, $du = 4x^3\,dx$. Then

$$\int_0^\infty x^3e^{-x^4}\,dx = \frac{1}{4}\int_0^\infty e^{-u}\,dx = -\frac{1}{4}\,e^{-u}\Bigg|_0^\infty = \frac{1}{4}.$$

Similarly,

$$\int_{-\infty}^{0} x^3 e^{-x^4}\,dx = -\frac{1}{4}\,e^{-u}\,\Big|_{\infty}^{0} = -\frac{1}{4}.$$

Then

$$\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx = \frac{1}{4} - \frac{1}{4} = 0.$$

Remarks. *It is also possible to simply find the antiderivative and then evaluate between limits M and N, say, and then consider the independent limits when M goes to $+\infty$ and N goes to $-\infty$. In that situation,*

$$\int_{N}^{M} x^3 e^{-x^4}\,dx = -\frac{1}{4}\,e^{-M^4} + \frac{1}{4}\,e^{-N^4},$$

and then taking the limits when M, N go to infinity shows that the integral is zero.
An easier approach is to note that the function $x^3e^{-x^4}$ is odd. Then applying the substitution $u = -x$ we get

$$\int_{-\infty}^{0} x^3 e^{-x^4}\,dx = \int_{\infty}^{0} u^3 e^{-u^4}\,du = -\int_{0}^{\infty} u^3 e^{-u^4}\,du.$$

Then, if we first prove that $\int_0^\infty x^3e^{-x^4}\,dx$ is convergent,

$$\begin{aligned}\int_{-\infty}^{\infty} x^3 e^{-x^4}\,dx &= \int_{-\infty}^{0} x^3 e^{-x^4}\,dx + \int_{0}^{\infty} x^3 e^{-x^4}\,dx \\ &= -\int_{0}^{\infty} x^3 e^{-x^4}\,dx + \int_{0}^{\infty} x^3 e^{-x^4}\,dx = 0\end{aligned}$$

Comparison shows that the integral is convergent without calculating it: since $x^3e^{-x^2} \leq 1$ for every x,

$$\int_{0}^{\infty} x^3 e^{-x^4}\,dx = \int_{0}^{\infty} (x^3 e^{-x^2}) e^{-x^2}\,dx \leq \int_{0}^{\infty} e^{-x^2}\,dx < \infty.$$

(d) $\displaystyle\int_2^3 \frac{1}{\sqrt{3-x}}\,dx$

Answer. This integral is improper at 3, because the denominator becomes zero there. The antiderivative is easy to find with the substitution $u = 3 - x$: then

$$\begin{aligned}\int_2^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{a\to 3}\int_2^a \frac{1}{\sqrt{3-x}}\,dx = \lim_{a\to 3}\left(-2\sqrt{3-x}\right)\Big|_2^a \\ &= 2 - \lim_{a\to 3} 2\sqrt{3-a} = 2.\end{aligned}$$

The integral is then convergent, and its value is 2.

(e) $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

Answer. We need to find the antiderivative. One way is by parts:

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\,\ln x - \int \frac{2\sqrt{x}}{x}\,dx = 2\sqrt{x}\ln x - 2\int x^{-1/2}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x}$$

Another way is to write

$$\frac{\ln x}{\sqrt{x}} = \frac{2\ln\sqrt{x}}{\sqrt{x}}$$

and then use the substitution $u = \sqrt{x}$ (and the antiderivative of $\ln u$, which we have calculated in the past) to get

$$\begin{aligned}\int \frac{\ln x}{\sqrt{x}}\,dx &= 4\int \ln u\,du = 4(-u + u\ln u) = -4\sqrt{x} + 4\sqrt{x}\,\ln\sqrt{x} \\ &= -4\sqrt{x} + 2\sqrt{x}\,\ln x.\end{aligned}$$

Now we can attempt the improper integral:

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = -4\sqrt{x} + 2\sqrt{x}\,\ln x\,\Big|_0^1 = -4 - 2\lim_{a\to 0}\sqrt{x}\ln x = -4.$$

The limit can be easily calculated by L'Hôpital (it is of the form $-\infty/\infty$)

$$\lim_{a\to 0}\sqrt{x}\ln x = \lim_{a\to 0}\frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{a\to 0}\frac{\frac{1}{x}}{\frac{-1}{2x^{3/2}}} = \lim_{a\to 0}(-2\sqrt{x}) = 0.$$

2. Find the values of s for which the integral converges, and evaluate the integral for those values of s:

$$\int_e^\infty \frac{1}{x\,(\ln x)^s}\,dx.$$

Answer. First we find the antiderivative. We have $1/x$ in the integral, with the rest depending on $\ln x$. This suggests the substitution $u = \ln x$, and then we find that the antiderivative is: when $s \neq 1$, $\frac{(\ln x)^{1-s}}{1-s}$; and when $s = 1$, $\ln(\ln x)$. Then

$$\int_e^\infty \frac{1}{x\,(\ln x)^s}\,dx \;=\; \frac{(\ln x)^{1-s}}{1-s}\Bigg|_e^\infty = -\frac{1}{1-s} + \lim_{N\to\infty} \frac{(\ln N)^{1-s}}{1-s}$$

When $s < 1$, the exponent of the logarithm will be positive, and the limit will be $+\infty$ (the $(1-s)$ in the quotient is positive in that case). When $s > 1$, the exponent of the logarithm is negative, and so the limit is zero. And when $s = 1$, we have

$$\int_e^\infty \frac{1}{x\,(\ln x)}\,dx = \ln(\ln x)\Bigg|_e^\infty = \lim_{N\to\infty} \ln(\ln N) = +\infty.$$

In summary,

$$\int_e^\infty \frac{1}{x\,(\ln x)^s}\,dx = \begin{cases} +\infty & \text{if } s \leq 1 \\ \dfrac{1}{s-1} & \text{if } s > 1 \end{cases}$$

3. Use comparison to determine whether the integral is convergent or divergent.

   (a) $\displaystyle\int_0^\infty \frac{\arctan x}{3 + 2e^x}\,dx$

   Answer. The denominator is always positive and continuous, and the inverse tangent function is always continuous, so the integral is only improper at infinity. The behaviour of arctan at infinity is to approach $\pi/2$, so we can basically think of it as a constant. And the denominator is essentially $2e^x$, so we can think of our integrand as $e^{-x}/2$. This function we know has a convergent integral at infinity, so the guess for our integral is that it converges. Thus we need to find a function above $\frac{\arctan x}{3+2e^x}$ and such that its integral is convergent. The arctan poses no problem, because $\arctan x \le \pi/2$ for every x. It is also clear that $3 + 2e^x \ge 2e^x$. Then

$$\frac{\arctan x}{3+2e^x} \le \frac{\pi/2}{3+2e^x} \le \frac{\pi/2}{2e^x} = \frac{\pi}{4e^x}.$$

Then

$$\int_0^\infty \frac{\arctan x}{3+2e^x}\,dx \le \int_0^\infty \frac{\pi}{4e^x}\,dx = \frac{\pi}{4}\int_0^\infty e^{-x}\,dx = \frac{\pi}{4} < \infty.$$

By the Comparison Theorem, we conclude that $\displaystyle\int_0^\infty \frac{\arctan x}{3+2e^x}\,dx$ converges.

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x+1}}\,dx$

Answer. In this case, the integrand is basically $\frac{1}{\sqrt{x}}$, which we know has a divergent integral at infinity. So the idea is to compare with $1/\sqrt{x}$. The problem is that the inequality we have is

$$\frac{1}{\sqrt{x+1}} \le \frac{1}{\sqrt{x}},$$

which is not useful to show by comparison that our integral diverges. But since $x \ge 1$, we have that $\sqrt{x+x} \ge \sqrt{x+1}$. Then $\frac{1}{\sqrt{x+x}} \le \frac{1}{\sqrt{x+1}}$, and

$$\int_1^\infty \frac{1}{\sqrt{x+1}}\,dx \ge \int_1^\infty \frac{1}{\sqrt{x+x}}\,dx = \frac{1}{\sqrt{2}}\int_1^\infty \frac{1}{\sqrt{x}}\,dx = +\infty.$$

By the Comparison Theorem, $\displaystyle\int_1^\infty \frac{1}{\sqrt{x+1}}\,dx$ diverges.

Remarks. *It is usually possible to do comparison with different functions, and this is an example of that. For example, one could use that* $\sqrt{x+1} \le x+1$ *(because* $x+1 \ge 1$*) to compare with the integral of* $1/(x+1)$*, which is easy to calculate.*

In fact, the very integral in this exercise is easy to calculate (substitute $u = x+1$*), but the point here was to do comparison. It was still useful to calculate it to be sure that it diverges and then use comparison in the right way.*

(c) $\int_3^\infty \frac{3+2e^{-x}}{x}\,dx$

Answer. Once again this integral is only improper at infinity. It is tempting to look at e^{-x} and conclude that the integral converges. But fact is that e^{-x} goes to zero at infinity, and so the numerator near infinity is almost 3; this says that the integral is basically that of $3/x$, which we know diverges. And the comparison is easy, because the exponential is always positive, and so $3+2e^{-x} \geq 3$. As we are considering positive x, the inequality is preserved when dividing by x. So we have

$$\int_3^\infty \frac{3+2e^{-x}}{x}\,dx \geq \int_3^\infty \frac{3}{x}\,dx = +\infty.$$

The Comparison Theorem then tells us that $\int_3^\infty \frac{3+2e^{-x}}{x}\,dx$ diverges.