Math 111-002

Assignment # 9 - Answers

1. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n = \dfrac{3+5n^2}{4+n}$

   Answer. We have, for $n \geq 4$,

   $$a_n = \frac{3+5n^2}{4+n} \geq \frac{5n^2}{4+4} \xrightarrow[n\to\infty]{} \infty$$

   So $\lim_{n\to\infty} a_n = \infty$.

   (b) $a_n = 8.3 + (0.715)^n$

   Answer. Since $0 < 0.715 < 1$, we get $\lim_{n\to\infty}(0.715)^n = 0$. Then

   $$\lim_{n\to\infty} 8.3 + (0.715)^n = 8.3 + \lim_{n\to\infty}(0.715)^n = 8.3 + 0 = 8.3$$

   (c) $a_n = \dfrac{7^n}{100+8^n}$

   Answer. We have, since $0 < 7/8 < 1$,

   $$0 \leq \frac{7^n}{100+8^n} \leq \frac{7^n}{8^n} = \left(\frac{7}{8}\right)^n \xrightarrow[n\to\infty]{} 0.$$

   By the squeeze property, $\lim_{n\to\infty} \dfrac{7^n}{100+8^n} = 0$.

   (d) $a_n = \dfrac{7^n}{n+8^n}$ **Answer.** We can repeat the argument from above: since $0 < 7/8 < 1$,

   $$0 \leq \frac{7^n}{n+8^n} \leq \frac{7^n}{8^n} = \left(\frac{7}{8}\right)^n \xrightarrow[n\to\infty]{} 0.$$

   By the squeeze property, $\lim_{n\to\infty} \dfrac{7^n}{n+8^n} = 0$.

(e) $a_n = \cos\left(\dfrac{n\pi}{n+2}\right)$

Answer. First, note that

$$\lim_{n\to\infty} \frac{n}{n+2} = \lim_{n\to\infty} \frac{1}{1+2/n} = 1.$$

Then, because the cosine is continuous,

$$\lim_{n\to\infty} \cos\left(\frac{n\pi}{n+2}\right) = \cos\left(\lim_{n\to\infty} \frac{n\pi}{n+2}\right) = \cos\pi = -1.$$

(f) $a_n = \ln(n+2) - \ln n$

Answer. Here,

$$\ln(n+2) - \ln n = \ln\left(\frac{n+2}{n}\right) = \ln\left(1+\frac{2}{n}\right).$$

Then, using the continuity of ln as 1,

$$\lim_{n\to\infty} \ln(n+2) - \ln n = \lim_{n\to\infty} \ln\left(1+\frac{2}{n}\right) = \ln\left(\lim_{n\to\infty} 1+\frac{2}{n}\right) = \ln 1 = 0.$$

(g) $a_n = n^{1/n}$

Answer. We have $n^{1/n} = e^{\frac{1}{n}\ln n}$. Using L'Hôpital,

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

The, by the continuity of the exponential,

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^{\lim_{n\to\infty} \frac{1}{n}\ln n} = e^0 = 1.$$

(h) $a_n = \cos n\pi$

Answer. We have $\cos n\pi = (-1)^n$. So the values of the sequence constantly alternate between 1 and -1, and so the limit cannot exist.

(i) $a_n = \dfrac{(\ln n)^2}{n}$

Answer. Again we use L'Hôpital (twice). We have

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{\frac{2}{x}\ln x}{1} = \lim_{x\to\infty} \frac{2}{x}\ln x = 0.$$

Then, as the limit of the function is zero at infinity, we get

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x} = 0.$$

(j) $a_n = \arctan(\ln n)$
Answer. We have $\lim_{n\to\infty} \ln n = \infty$. So

$$\lim_{n\to\infty} \arctan(\ln n) = \lim_{m\to\infty} \arctan m = \frac{\pi}{2}.$$

(k) $a_n = \left(1 + \frac{3}{n}\right)^n$
Answer. Here

$$\left(1 + \frac{3}{n}\right)^n = \exp\left(n \ln\left(1 + \frac{3}{n}\right)\right).$$

Using L'Hôpital,

$$\begin{aligned}\lim_{n\to\infty} n\ln\left(1+\frac{3}{n}\right) &= \lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{1/x} \\ &= \lim_{x\to\infty} \frac{-\frac{3}{x^2}/\left(1+\frac{3}{x}\right)}{-1/x^2} \\ &= \lim_{x\to\infty} \frac{3}{\left(1+\frac{3}{x}\right)} = 3.\end{aligned}$$

By the continuity of the exponential,

$$\lim_{n\to\infty} \exp\left(n \to \infty n \ln\left(1+\frac{3}{n}\right)\right) = \exp\left(\lim_{n\to\infty} n\ln\left(1+\frac{3}{n}\right)\right) = \exp(3) = e^3.$$