Math 111-002
Assignment # 10 - Answers

1. Suppose that q is a real number with $|q| < 1$.

   (a) Use a trick similar to the one we used in class to find the sum of the geometric series to show that

   $$\sum_{k=M}^{N} q^k = \frac{q^M - q^{N+1}}{1-q}.$$

   Answer. Let $S = \sum_{k=M}^{N} q^k$. Then $qS = \sum_{k=M}^{N} q^{k+1}$ and $S - qS = q^M - q^{N+1}$, so solving for S we get

   $$\sum_{k=M}^{N} q^k = S = \frac{q^M - q^{N+1}}{1-q}.$$

   (b) Prove that

   $$\sum_{k=M}^{\infty} q^k = \frac{q^M}{1-q}.$$

   Answer. By definition and part (a),

   $$\sum_{k=M}^{\infty} q^k = \lim_{N\to\infty} \sum_{k=M}^{N} q^k = \lim_{N\to\infty} \frac{q^M - q^{N+1}}{1-q} = \frac{q^M}{1-q}.$$

   The last equality will be justified if we show that $\lim_{N\to\infty} q^{N+1} = 0$. When $0 < q < 1$, $q^{N+1} = \exp\{(N+1)\ln q\}$; the fact that $q < 1$ translates in $\ln q < 0$, and so the exponential goes to zero at $-\infty$. When $q < 0$, $-q > 0$ and so

   $$\lim_{N\to\infty} q^{N+1} = \lim_{N\to\infty} (-1)^{N+1}(-q)^{N+1} = 0.$$

2. The decimal notation we use to write numbers is based on powers of 10. For example, when we write 543.25, we really mean $5 \times 10^2 + 4 \times 10^1 + 3 \times 10^0 + 2 \times 10^{-1} + 5 \times 10^{-2}$. In particular, when we write $0.9999999\ldots$ we mean $9 \sum_{k=1}^{\infty} \frac{1}{10^k}$. Use this fact to prove that

(a) $0.33333\ldots = 1/3$;

Answer. We have

$$0.33333\ldots = \sum_{n=1}^{\infty} \frac{3}{10^n} = 3 \sum_{n=1}^{\infty} \left(\frac{1}{10}\right)^n = 3\,\frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{3}{9} = \frac{1}{3}.$$

(b) $0.9999999\ldots = 1$.

We have

$$0.9999999\ldots = 9 \sum_{k=1}^{\infty} \frac{1}{10^k} = 9\,\frac{\frac{1}{10}}{1-\frac{1}{10}} = \frac{9}{9} = 1.$$

3. Use a geometric series to write $0.87\overline{43}$ as a fraction.

Answer. We have

$$0.87\overline{43} = \frac{87}{100} + \sum_{n=2}^{\infty} \frac{43}{100^n} = \frac{87}{100} + 43 \sum_{n=2}^{\infty} \frac{1}{100^n}$$

$$= \frac{87}{100} + 43\,\frac{\frac{1}{100^2}}{1-\frac{1}{100}} = \frac{87}{100} + \frac{43}{9900}$$

$$= \frac{87 \times 9900 + 43 \times 100}{9900} = \frac{865600}{990000}$$

4. Determine if the series is convergent or divergent. If it converges, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-7)^{n-1}}{6^{n-1}}$

Answer. This is a geometric series with $r = -7/6$. As $|-7/6| = 7/6 > 1$, the series is not convergent (the general term of the series does not converge to zero as $n \to \infty$).

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{7^{n-1}}$

Answer. Again, this is a geometric series, but now $|r| = 6/7 < 1$, so the series is convergent. Thus

$$\sum_{n=1}^{\infty} \frac{(-7)^{n-1}}{6^{n-1}} = \sum_{n=1}^{\infty} \left(-\frac{6}{7}\right)^{n-1} = \sum_{n=0}^{\infty} \left(-\frac{6}{7}\right)^{n} = \frac{1}{1-(-\frac{6}{7})} = \frac{7}{13}.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$

Answer. This looks like a series that could telescope. We have

$$\frac{1}{n(n+2)} = \frac{1}{2n} - \frac{1}{2(n+2)}.$$

Then, looking at the partial sums,

$$\sum_{n=1}^{N} \frac{1}{n(n+2)} = \sum_{n=1}^{N} \frac{1}{2n} - \frac{1}{2(n+2)} = \sum_{n=1}^{N} \frac{1}{2n} - \sum_{n=1}^{N} \frac{1}{2(n+2)}$$

$$= \sum_{n=1}^{N} \frac{1}{2n} - \sum_{n=3}^{N+2} \frac{1}{2n} = \frac{1}{2} + \frac{1}{4} - \frac{1}{2(N+1)} - \frac{1}{2(N+2)}.$$

As the limit as $N \to \infty$ exists, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{N\to\infty} \frac{1}{2} + \frac{1}{4} - \frac{1}{2(N+1)} - \frac{1}{2(N+2)} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

Answer. The denominator factors as $n^2+4n+3 = (n+1)(n+3)$. Then, using the previous answer (and noting that the sum here will start on the second term, so we need to subtract the term with $n = 1$),

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \sum_{n=1}^{\infty} \frac{2}{(n+1)(n+3)} = 2\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$$

$$= 2\left(\frac{3}{4} - \frac{1}{3}\right) = \frac{5}{6}.$$

(e) $\sum_{n=1}^{\infty} \frac{3n}{n^2+1}$

Answer. We have

$$\frac{3n}{n^2+1} \geq \frac{3n}{n^2+n^2} = \frac{3}{2n} \geq \frac{3}{3n} = \frac{1}{n}.$$

As the harmonic series diverges, our series also diverges by comparison.