Math 111-002
Assignment # 11 – Answers

1. Determine if the series is convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}+3}{n^2}$

Answer. This series compares with the one with term $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$. We have

$$\lim_{n\to\infty} \frac{\dfrac{\sqrt{n}+3}{n^2}}{\dfrac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\sqrt{n}+3}{\sqrt{n}} = \lim_{n\to\infty} 1 + \frac{3}{\sqrt{n}} = 1,$$

so the series converges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. We could have also done the comparison directly: for $n \geq 9$, $3 \leq \sqrt{n}$; then

$$\sum_{n=9}^{\infty} \frac{\sqrt{n}+3}{n^2} \leq \sum_{n=9}^{\infty} \frac{\sqrt{n}+\sqrt{n}}{n^2} = \sum_{n=9}^{\infty} \frac{2}{n^{3/2}} < \infty.$$

So the series converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4-2}$

Answer. Here we notice that $\dfrac{n^3}{n^4-2}$ is roughly $\dfrac{n^3}{n^4} = \dfrac{1}{n}$. Starting with $n = 2$, we have $n^4 - 2 \leq n^4$, so $\dfrac{1}{n^4-2} \geq \dfrac{1}{n^4}$. Multiplying by n^3, positive, will preserve this inequality, so

$$\sum_{n=2}^{\infty} \frac{n^3}{n^4-2} \geq \sum_{n=2}^{\infty} \frac{n^3}{n^4} = \sum_{n=2}^{\infty} \frac{1}{n} = \infty.$$

So the series diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^3}{n^4-2}$

Answer. This is an alternating series. To apply Leibnitz criterion, we need to check that the absolute values of the terms are decreasing and that they go to zero; the latter is obvious. If we consider the function $f(x) = x^3/(x^4-2)$, we note that it is discontinuous at $x = 2^{1/4}$, but for $x > 2^{1/4}$, its derivative is

$$f'(x) = \frac{3x^2(x^4-2) - x^3 4x^3}{(x^4-2)^2} = \frac{-x^6 - 6x^2}{(x^4-2)^2} < 0.$$

The derivative is always negative because the numerator is negative and the denominator is positive. So f is decreasing the Leibnitz criterion applies: the series is convergent.

(d) $\displaystyle\sum_{n=3}^{\infty} \frac{2n+1}{n^2-2n}$

Answer. This series compares with the harmonic series:

$$\lim_{n\to\infty} \frac{\dfrac{2n+1}{n^2-n}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{(2n+1)n}{n^2-n} = \lim_{n\to\infty} \frac{2n+1}{n-1} = 2,$$

so the series diverges by comparison with the harmonic series.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n+n}{8^n-n^3}$

Answer. We can compare this series with the geometric series with $q = 5/8$. We have

$$\lim_{n\to\infty} \frac{\dfrac{5^n+n}{8^n-n^3}}{\dfrac{5^n}{8^n}} = \lim_{n\to\infty} \frac{5^n+n}{5^n} \frac{8^n}{8^n-n^3} = 1,$$

so the series converges by comparison with $\displaystyle\sum_{n=3}^{\infty} \left(\frac{5}{8}\right)^n$. We could have also applied the ratio test:

$$\frac{\dfrac{5^{n+1}+n+1}{8^{n+1}-(n+1)^3}}{\dfrac{5^n+n}{8^n-n^3}} = \frac{5^{n+1}+n+1}{5^n+n} \frac{8^n-n^3}{8^{n+1}-(n+1)^3} \xrightarrow[n\to\infty]{} \frac{5}{8} < 1.$$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{8^n+n}{5^n-n^3}$

Answer. The computations here are basically the same as above: we get that the series compares with the geometric series with $q = 8/5$, so it diverges. Or by the ratio test, the quotient $\frac{a_{n+1}}{a_n}$ goes to $\frac{8}{5} > 1$ and again we conclude that the series diverges.

(g) $\displaystyle\sum_{n=1}^{\infty} n^{-n}$

Answer. This is a nice candidate for the root test: we have

$$\lim_{n\to\infty} (n^{-n})^{1/n} = \lim_{n\to\infty} n^{-1} = 0,$$

so the series converges. We could have also used comparison: when $n \geq 2$,

$$n^{-n} = \frac{1}{n^n} \leq \frac{1}{n^2},$$

and we get convergence by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}}$

Answer. We can compare directly with the harmonic series:

$$\lim_{n\to\infty} \frac{\dfrac{1}{\sqrt{n^2+n}}}{\dfrac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}} = 1,$$

so we conclude that the series diverges by comparison with the harmonic series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+n}}$

Answer. Now we are dealing with an alternating series, so we only need to check whether the absolute values decrease to zero. We already know they go to zero, by the previous question. As for "decreasing", if we consider $f(x) = 1/\sqrt{x^2+x}$, its derivative is

$$f'(x) = -\frac{1}{2}\,\frac{2x+1}{(x^2+x)^{3/2}},$$

which is negative for all $x \geq 1$ (and for others too, but we only care about $x \geq 1$). Then, by Leibnitz's criterion, the series converges.

(j) $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1}+\sqrt{n})$

Answer. Here

$$|a_n| = \sqrt{n+1}+\sqrt{n} \xrightarrow[n\to\infty]{} \infty,$$

so the series diverges.

(k) $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1}-\sqrt{n})$

Answer. We have

$$\sqrt{n+1}-\sqrt{n} = (\sqrt{n+1}-\sqrt{n})\,\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

The limit at infinity is clearly zero. And it is decreasing:

$$|a_{n+1}| = \frac{1}{\sqrt{(n+1)+1}+\sqrt{n+1}} \leq \frac{1}{\sqrt{n}+\sqrt{n+1}} = |a_n|$$

(or one can check the derivative of the corresponding function). Then, by Leibnitz's criterion, the series is convergent.

2. Estimate $\displaystyle\sum_{n=1}^{\infty} \frac{2}{(4n+2)^5}$ with six correct decimals.

Answer. The series is convergent, by comparison with $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^5} < \infty$. By the Integral Test,

$$\sum_{n=N+1}^{\infty}\frac{2}{(4n+2)^5} \leq \int_N^{\infty}\frac{2}{(4x+2)^5}\,dx = \frac{1}{2}\int_{4N+2}^{\infty} u^{-5}\,du = -\frac{1}{8}\frac{1}{u^4}\bigg|_{4N+2}^{\infty} = \frac{1}{8(4N+2)^4}.$$

So we want

$$\frac{1}{8(4N+2)^4} \leq \frac{1}{1000000},$$

or $(4N+2)^4 \geq 1000000/8 = 125000$. So

$$N \geq \frac{125000^{1/4}-2}{4} \simeq 4.2.$$

That is, five terms will suffice. So

$$\sum_{n=1}^{\infty}\frac{2}{(4n+2)^5} \sim \sum_{n=1}^{5}\frac{2}{(4n+2)^5} = 2\left(\frac{1}{6^5}+\frac{1}{10^5}+\frac{1}{14^5}+\frac{1}{18^5}+\frac{1}{22^5}\right)$$

$$= \frac{2256575620655443}{7991644884334650000} \sim 0.000282\ldots$$

3. Estimate $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\,2^n}$ with three correct decimals.

Answer. This is an alternating series that satisfies the conditions in Leibnitz's criterion. So the $(N+1)^{\text{th}}$ term is a bound on the error of the approximation by the first N terms. That is, we need

$$\frac{1}{(N+1)^2\,2^{N+1}} \leq \frac{1}{1000}.$$

Already when $N=5$, we get $1/((5+1)^2\,2^{5+1}) = 1/1600$. Adding the first five terms we get

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\,2^n} \simeq \sum_{n=1}^{5}\frac{(-1)^{n+1}}{n^2\,2^n} = -\frac{1}{2}+\frac{1}{16}-\frac{1}{72}+\frac{1}{256}-\frac{1}{800} = -\frac{25847}{57600} \sim -0449.$$

4. Find the values of p for which the series is convergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(\ln n)^p}$

Answer. Since the function $f(x) = 1/(x(\ln x)^p)$ is decreasing, the series will converge precisely when the associated improper integral converges. We have

$$\int_2^{\infty}\frac{1}{x(\ln x)^p}\,dx = \int_{\ln 2}^{\infty}\frac{1}{u^p}\,du,$$

which will be convergent precisely when $p>1$. So the series converges for $p>1$.

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n (\ln(\ln n))^p}$

Answer. The situation is similar to the previous question, but now the improper integral is

$$\int_2^{\infty} \frac{1}{x\ \ln x(\ln(\ln x))^p}\, dx = \int_{\ln 2}^{\infty} \frac{1}{u\,(\ln u)^p}\, du = \int_{\ln(\ln 2)}^{\infty} \frac{1}{v^p}\, dv,$$

so again convergence will occur precisely when $p > 1$.

(c) $\displaystyle\sum_{n=3}^{\infty} \frac{\ln n}{n^p}$

Answer. When $p = 1$, the integral test gives us

$$\int_2^{\infty} \frac{\ln x}{x}\, dx = \int_{\ln 2}^{\infty} u\, du = \infty,$$

so the series diverges. Now for $p < 1$ we can use comparison to deduce that the series diverges. For $p > 1$, we can write $p = 1 + \frac{p-1}{2} + \frac{p-1}{2}$. Then we can show (using L'Hôpital) that

$$\lim_{n \to \infty} \frac{\ln n}{n^{\frac{p-1}{2}}} = 0,$$

so $\ln n / n^{\frac{p-1}{2}} < 1$ for n big enough, say $n \geq K$. Then

$$\sum_{n=K}^{\infty} \frac{\ln n}{n^p} = \sum_{n=K}^{\infty} \frac{\ln n}{n^{\frac{p-1}{2}}} \frac{1}{n^{1+\frac{p-1}{2}}} \leq \sum_{n=K}^{\infty} \frac{1}{n^{1+\frac{p-1}{2}}} < \infty$$

by (4a).

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$

Answer. Now we have an alternating series. For any $p > 0$, $\lim_{n\to\infty} \frac{1}{n^p} = 0$, and since $\frac{1}{n^p}$ is decreasing on n, by Leibnitz's criterion the series converges. For $p \leq 0$ the general term of the series does not converge to zero, so the series does not converge.