Math 111-002
Assignment # 12 – Answers

1. Look at the trick we used to find power series representations of $\ln(1+x)$ and $\arctan x$. Use an analog trick – but with differentiation instead of integration – to find a power series representation of
$$f(x) = \frac{1}{(1-x)^2}, \quad |x| < 1.$$

Answer. We know that, since $|x| < 1$,
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n;$$
if we differentiate this equality (which we can do because of the theorem that guarantees that power series can be differentiated term by term within their interval of convergence) we get
$$f(x) = \frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=0}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n$$

Remark. *The last equality is simply a change of index, from n to $n+1$; if you don't see this clearly, just write the first few terms of both sums to check they are equal.*

2. Find power series representations for

(a) $\dfrac{x^3}{(1-x)^2}$

Answer. Note that, if we write $g(x) = \frac{x^3}{(1-x)^2}$, then $g(x) = x^3 f(x)$ with f as in the previous question. For $|x| < 1$,
$$g(x) = x^3 \sum_{n=0}^{\infty} (n+1)x^n = \sum_{n=0}^{\infty} (n+1)x^{n+3} = \sum_{n=3}^{\infty} (n-2)x^n$$

(b) $\dfrac{1}{(1-x)^3}$

Answer. For $h(x) = \frac{1}{(1-x)^3}$, note that $f'(x) = \frac{2}{(1-x)^3}$, so $h(x) = \frac{1}{2} f'(x)$. Then
$$h(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} (n+1)x^n\right)' = \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^{n-1} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n$$

(c) $\left(\dfrac{x}{2-x}\right)^3$.

Answer. Let $k(x) = \left(\frac{x}{2-x}\right)^3$. We can write $k(x) = \frac{x^3}{(2-x)^3}$. We have

$$\frac{1}{(2-x)^3} = \frac{1}{2}\left(\frac{1}{(2-x)^2}\right)' = \frac{1}{2}\left(\frac{1}{2-x}\right)''.$$

Using a geometric series trick,

$$\frac{1}{2-x} = \frac{1}{2}\frac{1}{1-\frac{x}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}, \qquad -2 < x < 2$$

(for the radius of convergence, note that we need $|x/2| < 1$). So we have

$$\frac{x^3}{(2-x)^3} = x^3\frac{1}{(2-x)^3} = \frac{x^3}{2}\left(\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}\right)'' = \frac{x^3}{4}\sum_{n=0}^{\infty}\frac{n(n-1)x^{n-2}}{2^n}$$

$$= \sum_{n=0}^{\infty}\frac{n(n-1)x^{n+1}}{2^{n+2}} = \sum_{n=3}^{\infty}\frac{(n-1)(n-2)x^n}{2^{n+1}}$$

3. Find the radius and interval of convergence, and an explicit formula for

(a) $\displaystyle\sum_{n=1}^{\infty} nx^n$

Answer. We found above that

$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}, \qquad -1 < x < 1.$$

Then

$$\sum_{n=0}^{\infty} nx^n = \sum_{n=0}^{\infty}[(n+1)-1]x^n = \sum_{n=0}^{\infty} nx^n - \sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2} - \frac{1}{1-x}$$

$$= \frac{1-(1-x)}{(1-x)^2} = \frac{x}{(1-x)^2}.$$

To confirm the radius of convergence, we have

$$\lim_{n\to\infty}\frac{n}{n+1} = 1,$$

so the radius of convergence is 1 and the interval of convergence is $(-1, 1)$. The series does not convergence at $x = 1$ nor at $x = -1$.

(b) $\displaystyle\sum_{n=1}^{\infty} n^2(x-3)^n$

Answer. Using part (3a) we have

$$\sum_{n=1}^{\infty} n(x-3)^n = \frac{x-3}{(1-(x-3))^2} = \frac{x-3}{(4-x)^2},$$

for $-1 < x-3 < 1$, that is $2 < x < 4$. Taking derivatives,

$$\sum_{n=1}^{\infty} n^2(x-3)^{n-1} = \left(\sum_{n=1}^{\infty} n(x-3)^n\right)' = \left(\frac{x-3}{(4-x)^2}\right)' = \frac{x-2}{(4-x)^3}$$

Then

$$\sum_{n=1}^{\infty} n^2(x-3)^n = (x-3)\sum_{n=1}^{\infty} n^2(x-3)^{n-1} = \frac{(x-3)(x-2)}{(4-x)^3}, \quad 2 < x < 4.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Answer. We have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Taking antiderivatives,

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

4. Use a power series to find $\displaystyle\int_0^{1/2} \arctan\frac{x}{2}\,dx$ with three good decimals.

Answer. We know that

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad -1 < x < 1.$$

Then

$$\int_0^{1/2} \arctan\frac{x}{2}\,dx = \int_0^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+1}}{2n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)2^{2n+1}} \int_0^{1/2} x^{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)2^{2n+1}} \left.\frac{x^{2n+2}}{2n+2}\right|_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)\,2^{2n+1}2^{2n+2}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n+1)\,2^{4n+4}}$$

As this is an alternating series, a bound for the approximation of the sum up to N is given by

$$\left|\frac{(-1)^{N+1}}{(2(N+1)+1)(N+1+1)\,2^{4(N+1)+4}}\right| = \frac{1}{(2N+3)(N+2)\,2^{4N+8}}.$$

We want this last number to be less than 1/1000. This is comfortably achieved when $N = 0$. So

$$\int_0^{1/2} \arctan\frac{x}{2}\,dx \approx \frac{1}{2^4} = \frac{1}{16} = 0.0625.$$

5. Find the Taylor polynomial of each function at the given point a.

   (a) $f(x) = \cos x$, $a = \pi/2$;
   Answer. The fourth derivative of $f(x) = \cos x$ is again $\cos x$, and the first derivatives are

   $$f^{(0)}(x) = \cos x,\ \ f^{(1)}(x) = -\sin x,\ \ f^{(2)}(x) = -\cos x,\ \ f^{(3)}(x) = \sin x,\ \ f^{(4)}(x) = \cos x.$$

   Since $a = \pi/2$,

   $$f^{(0)}(a) = 0,\quad f^{(1)}(a) = -1,\quad f^{(2)}(a) = 0,\quad f^{(3)}(a) = 1,\quad f^{(4)}(a) = 0.$$

   So

   $$\begin{aligned}\cos x &= 0 - (x - \frac{\pi}{2}) + 0 + \frac{(x-\frac{\pi}{2})^3}{3!} + 0 - \frac{(x-\frac{\pi}{2})^5}{5!} + 0 + \cdots \\ &= \sum_{k=1}^{n} \frac{(-1)^k (x-\frac{\pi}{2})^{2k-1}}{(2k-1)!} + R_n(x),\end{aligned}$$

   where $R_n(x) = \frac{(-1)^{n+1} f^{(2n+1)}(c)(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$, with c between x and $\pi/2$, and $|f^{2n+1}(c)| \leq 1$.

   (b) $f(x) = e^{-x}$, $a = 0$.
   Answer. Since we already know the Taylor polynomial for e^x at $a = 0$, we can simply replace x with $-x$. So, since

   $$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^c x^{n+1}}{(n+1)!}, \quad c \text{ between } 0 \text{ and } x$$

   we have, writing $(-x)^k = (-1)^k x^k$,

   $$e^{-x} = \sum_{k=0}^{n} \frac{(-1)^k x^k}{k!} + \frac{(-1)^{n+1} e^c x^{n+1}}{(n+1)!}, \quad c \text{ between } 0 \text{ and } -x.$$

6. Use the Taylor polynomials found in class to calculate the following:

(a) The number e, with 6 good decimals.

Answer. We know that $e = e^1$, so

$$e = \sum_{k=0}^{n} \frac{1^k}{k!} + \frac{e^c 1^{n+1}}{(n+1)!}, \quad c \text{ between } 0 \text{ and } 1,$$

that is

$$e = \sum_{k=0}^{n} \frac{1}{k!} + \frac{e^c}{(n+1)!}, \quad c \text{ between } 0 \text{ and } 1.$$

To get 6 good decimals, we need the remainder to be less than 10^{-6}. We can use that $e < 3$, and since $c \leq 1$, $e^c \leq e < 3$; so

$$R_n(1) = \frac{e^c}{(n+1)!} \leq \frac{3}{(n+1)!}.$$

We will have $R_n(1) < 10^{-6}$ if $(n+1)! > 3\times 10^6$. We have that $10! = 3628800 > 3\times 10^6$, so $n = 9$ will be enough. Then

$$\begin{aligned} e &\simeq 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} \\ &= 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \frac{1}{40320} + \frac{1}{362880} \\ &= 2.718282. \end{aligned}$$

(b) The number $1/e$ with 6 good decimals.

Answer. Now we use that $1/e = e^{-1}$. Note that the the only difference between the reminder for e and the reminder for e^{-1} is eventually the sign. So the same estimate from (a) is good, and we only need to add until $n = 9$ (in fact the estimate is better here, because we don't need the 3: as $c \in [-1, 0]$, here we have $R_n(-1) \leq 1/(n+1)!$; in this particular case, it doesn't make a difference in the value of n, but in general it could). So

$$\begin{aligned} \frac{1}{e} &\simeq 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} - \frac{1}{8!} + \frac{1}{9!} \\ &= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} + \frac{1}{5040} - \frac{1}{40320} + \frac{1}{362880} \\ &= 0.367879. \end{aligned}$$

(c) $\arctan \frac{1}{2}$ with 7 good decimals.

Answer. Because of the way we found the Taylor series for arctan, we don't have an easy way to write the remainder. But we can use the fact that it's series is an alternating series, and so the remainder of the Taylor polynomial is less (in absolute value) than the next term in the series. So, since

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1},$$

we get that

$$|R_n(x)| \le \left|\frac{(-1)^{n+1}x^{2n+3}}{2n+3}\right| = \frac{x^{2n+3}}{2n+3}.$$

Estimating the reminder for $x = 1/2$, we get

$$|R_n(1/2) = \left|\frac{(-1)^{n+1}(1/2)^{2n+3}}{2n+3}\right| = \frac{1}{(2n+3)2^{2n+3}}.$$

We need this to be less than 10^{-7}. This can be achieved with $n = 9$ ($n = 8$ almost does, but it gets slightly above 10^{-7}). Then

$$\begin{aligned}
\arctan\frac{1}{2} &\simeq \frac{1}{2} - \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^7}{7} + \frac{\left(\frac{1}{2}\right)^9}{9} - \frac{\left(\frac{1}{2}\right)^{11}}{11} + \frac{\left(\frac{1}{2}\right)^{13}}{13} - \frac{\left(\frac{1}{2}\right)^{15}}{15} + \frac{\left(\frac{1}{2}\right)^{17}}{17} - \frac{\left(\frac{1}{2}\right)^{19}}{19} \\
&= \frac{1}{2} - \frac{1}{2^3 \times 3} + \frac{1}{2^5 \times 5} - \frac{1}{2^7 \times 7} + \frac{1}{2^9 \times 9} - \frac{1}{2^{11} \times 11} + \frac{1}{2^{13} \times 13} - \frac{1}{2^{15} \times 15} \\
&\quad + \frac{1}{2^{17} \times 17} - \frac{1}{2^{19} \times 19} \\
&= 0.4636476
\end{aligned}$$