University of Regina
Department of Mathematics and Statistics
Math 111-002
Midterm 1 – Answers

[10 Marks] 1. Find $(f^{-1})'(0)$, where $f(x) = x^5 + 5x - 6$.

Answer. We will use the Inverse Function Theorem, which says that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

The first thing we need is to determine $f^{-1}(0)$. By inspection, we can check that $f(1) = 0$, so that $f^{-1}(0) = 1$. Next, we need the derivative of f, $f'(x) = 5x^4 + 5$. Then

$$(f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{5+5} = \frac{1}{10}.$$

[10 Marks] 2. Evaluate the integral

$$\int_0^1 \frac{e^x}{(e^x+1)\,\ln(e^x+1)}\,dx.$$

(Hint: use substitution)

Answer. The easiest way to do the integral is to notice that $e^x/(e^x+1)$ is the derivative of $\ln(e^x+1)$. So, if we put $u = \ln(e^x+1)$, $du = \frac{e^x}{e^x+1}\,dx$, and

$$\int_0^1 \frac{e^x}{(e^x+1)\,\ln(e^x+1)}\,dx = \int_{x=0}^{x=1} \frac{1}{u}\,du = \ln u\Big|_{x=0}^{x=1} = \ln(\ln(e^x+1))\Big|_0^1$$

$$= \ln(\ln(e+1)) - \ln(\ln(2)) = \ln\frac{\ln(e+1)}{\ln 2}.$$

It was not necessary to notice this, though. If we go with $u = e^x + 1$, then $du = e^x, dx$, and

$$\int_0^1 \frac{e^x}{(e^x+1)\,\ln(e^x+1)}\,dx = \int_2^{e+1} \frac{1}{u\,\ln u}\,du.$$

This last integral was in assignment 3, and it can be done by substitution : with $v = \ln u$, we have $dv = \frac{1}{u}\,du$, so

$$\int_2^{e+1} \frac{1}{u\,\ln u}\,du = \int_{\ln 2}^{\ln(e+1)} \frac{1}{v}\,dv = \ln v\Big|_{\ln 2}^{\ln(e+1)} = \ln(\ln(e+1)) - \ln(\ln 2).$$

[10 Marks] 3. Find the limit

(a) $\lim_{x\to 0} \dfrac{\arctan(x) - x}{x^3}$.

Answer. Both numerator and denominator are differentiable, and the quotient is of the form 0/0 at 0. So we apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\arctan(x) - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1}{3x^2} = \lim_{x\to 0} \frac{1-(1+x^2)}{3x^2(1+x^2)}$$

$$= \lim_{x\to 0} \frac{-x^2}{3x^2(1+x^2)} = \lim_{x\to 0} \frac{-1}{3(1+x^2)} = -\frac{1}{3}.$$

(b) $\lim_{x\to 0} (2x + \cos x)^{1/x}$.

Answer. By definition of exponentiation,

$$(2x + \cos x)^{1/x} = e^{\frac{1}{x}\ln(2x+\cos x)}.$$

As the exponential is continuous, we may take the limit of the exponent. The exponent is a quotient, where both numerator and denominator are zero when $x = 0$ (note that $\ln \cos 0 = \ln 1 = 0$). Then we apply L'Hôpital's rule:

$$\lim_{x\to 0} \frac{\ln(2x+\cos x)}{x} = \lim_{x\to 0} \frac{\frac{2-\sin x}{2x+\cos x}}{1} = \lim_{x\to 0} \frac{2-\sin x}{2x+\cos x} = \frac{2-0}{0+\cos 0} = 2.$$

As the last limit exists, L'ôpital's rule applies. Then

$$\lim_{x\to 0} (2x+\cos x)^{1/x} = \exp(\lim_{x\to 0} \frac{\ln(2x+\cos x)}{x}) = \exp(2) = e^2.$$

Comments. *In part a), many students didn't simplify the expression after the first application of L'Hôpital. This led them to apply it two more times, with increasingly complicated derivatives and increased chances for errors.*

A very common mistake in part a) was not to mention the use of L'Hôpital, and in some cases not even write the limits. Please be careful in writing the steps you make and in justifying them.

The most common mistake in part b) was to—correctly—calculate the limit of the logarithm of the expression. This limit was 2, and most calculated it correctly. But it was not the answer to the question! Again, please make sure you justify all your steps, and that you write the final answer carefully.