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University of Regina
Department of Mathematics and Statistics
Math 111-002
Midterm 2 – Answers

[10 Marks] 1. Evaluate the integral $\int_0^1 x \arctan x \, dx$.

Answer. *(This was question 1c in Assignment 6)* By Parts,

$$\begin{aligned}\int_0^1 x \, \arctan(x) \, dx &= \frac{x^2}{2} \arctan x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2} dx = \frac{\pi}{8} - \frac{1}{2}\int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx \\ &= \frac{\pi}{8} - \frac{1}{2}\left(x - \arctan x\right)\Big|_0^1 = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

[10 Marks] 2. Evaluate the integral $\int_0^1 \frac{x^3}{\sqrt{2+x^2}} dx$

Answer. *(This was question 1e in Assignment 6)* We start with the substitution $u = 2 + x^2$, and we get

$$\begin{aligned}\int_0^1 \frac{x^3}{\sqrt{2+x^2}} dt &= \frac{1}{2}\int_2^3 \frac{u-2}{\sqrt{u}} du = \frac{1}{2}\int_2^3 (u^{1/2} - 2u^{-1/2}) \, du \\ &= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 4u^{1/2}\right)\Big|_2^3 = -\sqrt{3} + \frac{4}{3}\sqrt{2}.\end{aligned}$$

It is also possible to do this integral by trigonometric substitution (and more work!): if we let $x = \sqrt{2}\tan v$, then $dt = \sqrt{2}\sec^2 v \, dv$, and

$$\begin{aligned}\int_0^1 \frac{x^3}{\sqrt{2+x^2}} dt &= \int_{t=0}^{t=1} \frac{2\tan^3 v}{\sec v}\sqrt{2}\sec^2 v \, dv = 2\sqrt{2}\int_{t=0}^{t=1} \tan^3 v \, \sec v \, dv \\ &\quad \text{(now we substitute } w = \sec v\text{, with } dw = \tan v \, \sec v \, dv\text{)} \\ &= 2\sqrt{2}\int_{t=0}^{t=1} (w^2 - 1) dw = 2\sqrt{2}\left(\frac{w^3}{3} - w\right)\Big|_{t=0}^{t=1} \\ &= 2\sqrt{2}\left(\frac{\sec^3(\arctan\frac{t}{\sqrt{2}})}{3} - \sec(\arctan\frac{t}{\sqrt{2}})\right)\Big|_0^1 \\ &\quad \text{(now you need some trigonometry to determine that} \\ &\quad \sec(\arctan\tfrac{1}{\sqrt{2}}) = \tfrac{\sqrt{3}}{\sqrt{2}}\text{: this comes from the square triangle} \\ &\quad \text{with sides } 1, \sqrt{2}, \sqrt{3}\text{)} \\ &= 2\sqrt{2}\left(\frac{(\sqrt{3})^3}{3(\sqrt{2})^3} - \frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{3} + 1\right) = \sqrt{3} - 2\sqrt{3} + \frac{4}{3}\sqrt{2} \\ &= -\sqrt{3} + \frac{4}{3}\sqrt{2}.\end{aligned}$$

Yet a third way is to do the integral by parts: indeed, if $f(x) = x^2$, $g(x) = \sqrt{2+x^2}$, then $g'(x) = x/\sqrt{2+x^2}$ and

$$\begin{aligned}\int_0^1 \frac{x^3}{\sqrt{2+x^2}}\,dt &= \int_0^1 x^2\,\frac{x}{\sqrt{2+x^2}}\,dx = \int_0^1 f(x)\,g'(x)\,dx \\ &= x^2\sqrt{2+x^2}\,\Big|_0^1 - \int_0^1 2x\sqrt{2+x^2}\,dx \\ &= \sqrt{3} - \frac{2}{3}\,(2+x^2)^{3/2}\,\Big|_0^1 = \sqrt{3} - \frac{2}{3}\,3^{3/2} + \frac{2}{3}\,2^{3/2} \\ &= \sqrt{3} - 2\sqrt{3} - \frac{4}{3}\sqrt{2} = -\sqrt{3} + \frac{4}{3}\sqrt{2}.\end{aligned}$$

[10 Marks] 3. Find $\displaystyle\int \frac{2x+1}{(x+1)(x^2+1)}\,dx$.

Answer. *(This was question 2.4.1.b in the Lab Manual)* We need to do partial fractions. So we set up

$$\begin{aligned}\frac{2x+1}{(x+1)(x^2+1)} &= \frac{A}{x+1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)} \\ &= \frac{(A+B)x^2+(B+C)x+A+C}{(x+1)(x^2+1)}\end{aligned}$$

So $A+B=0$, $B+C=2$, $A+C=1$. From $B=-A$, we reduce this to $C-A=2$, $C+A=1$. Adding and then subtracting the two equations, $2C=3$, $2A=-1$. So $A=-1/2$, $B=1/2$, $C=3/2$. Then

$$\begin{aligned}\int \frac{2x+1}{(x+1)(x^2+1)}\,dx &= \int \left(-\frac{1}{2(x+1)} + \frac{x+3}{2(x^2+1)}\right)dx \\ &= -\frac{1}{2}\ln(x+1) + \frac{1}{2}\int \frac{x}{x^2+1} + \frac{3}{2}\int \frac{1}{x^2+1}\,dx \\ &\quad \textit{(using } u = x^2+1\textit{)} \\ &= -\frac{1}{2}\ln(x+1) + \frac{1}{4}\ln(x^2+1) + \frac{3}{2}\arctan x.\end{aligned}$$

[10 Marks] 4. Determine if $\displaystyle\int_0^\infty \frac{\arctan x}{5+2e^x}\,dx$ converges or diverges.

Answer. *(This was question 3a in Assignment 8)* Since $\arctan x \le \frac{\pi}{2}$ for all x, we get by comparison that

$$\int_0^\infty \frac{\arctan x}{5+2e^x}\,dx \le \frac{\pi}{2}\int_0^\infty \frac{1}{5+2e^x}\,dx \le \frac{\pi}{2}\int_0^\infty \frac{1}{2e^x}\,dx = \frac{\pi}{4}\,\left(-e^{-x}\right)\Big|_0^\infty = \frac{\pi}{4} < \infty.$$

So the integral converges.