Math 221-001 201710
Assignment # 2 - Answers

1. Let A, B, C be subsets of $\mathcal{U}$. Prove

   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;

   Answer. Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$; if $x \in B$, then $x \in A \cap B$; otherwise, $x \in C$ and so $x \in A \cap C$. In any case, $x \in (A \cap B) \cup (A \cap C)$. This proves that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

   If $x \in (A \cap B) \cup (A \cap C)$, then either $x \in (A \cap B)$ or $x \in (A \cap C)$ hold (possibly both); if $x \in (A \cap B)$, then $x \in A$ and $x \in B$, so $x \in A \cap (B \cup C)$; similarly, if $x \in (A \cap C)$, we deduce that $x \in A \cap (B \cap C)$.

   (b) $(A \cap B)^c = A^c \cup B^c$;

   Answer. We have that $x \in (A \cap B)^c$ if and only if $x \notin A \cap B$; this happens if and only if $x \notin A$ or $x \notin B$ (this is the logical "or", where both possibilities are allowed to happen together); this is the same as saying that $x \in A^c$ or $x \in B^c$, and this in turn is saying that $x \in A^c \cup B^c$.

   We conclude that $x \in (A \cap B)^c \Leftrightarrow x \in A^c \cup B^c$, so $(A \cap B)^c = A^c \cup B^c$.

2. Indicate "true" or "false" in each of the following assertions. Explain.

   (a) $\emptyset \in \{\emptyset, \{\emptyset\}\}$

   Answer. TRUE, because $\emptyset$ appears in the list of elements at the right (the first one).

   (b) $\emptyset \subset \{\emptyset, \{\emptyset\}\}$

   Answer. TRUE, because the empty set is a subset of any set.

   (c) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

   Answer. TRUE, because the set $\{\emptyset\}$ has $\emptyset$ as an element, which is also an element of the set at the right (the first one in the list).

   (d) $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$

   Answer. TRUE, because $\{\emptyset\}$ is an element of the set at the right (the second one).

   (e) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

   Answer. TRUE, because the set $\{\{\emptyset\}\}$ has one element, $\{\emptyset\}$, which is an element of the set at the right (the second in the list).

   (f) $\{a, b\} \in \{\{a, b, c\}, \{a, c\}, a, b\}$

   Answer. FALSE, because $\{a, b\}$ is not an element of the set at the right (it is not in the list).

   (g) $\{a, b\} \subset \{\{a, b, c\}, \{a, c\}, a, b\}$

   Answer. TRUE, because both a and b are elements of the set at the right (the two last in the list).

(h) $\emptyset \in \{\{a, b, c\}, \{a, c\}, a, b\}$

Answer. FALSE, because the empty set is not an element of that set.

(i) $\emptyset \subset \{\{a, b, c\}, \{a, c\}, a, b\}$

Answer. TRUE, the empty set is a subset of any set.

(j) $\emptyset \in \mathcal{P}(\{1, 2\})$.

Answer. TRUE. The empty set is included in any set, so the empty set is an element of the power set of any set.

3. Let A, B, C be sets. Determine if the following assertions are true or false. Provide a proof or a counterexample in each case.

(a) $(A \cap B) \subset (B \cap C) \Rightarrow A \subset B$.

Answer. FALSE. Take A, C to be any nonempty sets, and $B = \emptyset$. There are infinitely many other possible counterexamples.

(b) $(A \cup B) \subset (B \cup C) \Rightarrow A \subset B$.

Answer. FALSE. Again, choose $B = \emptyset$, and $A = C$ any nonempty set.

(c) $\mathcal{P}(C) \setminus \mathcal{P}(B) \subset \mathcal{P}(B \setminus C)$.

Answer. FALSE. Take C to be any nonempty set, and $B = \emptyset$, then the set in the left is $\mathcal{P}(C) - \{\emptyset\}$, which is nonempty and has elements different from $\emptyset$, while the set in the right is $\{\emptyset\}$.

(d) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

Answer. Let $X \in \mathcal{P}(A \cap B)$; then $X \subset A \cap B$. So we have $X \subset A$ and $X \subset B$, that is $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, so $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. This proves that $\mathcal{P}(A \cap B) \subset \mathcal{P}(A) \cap \mathcal{P}(B)$. Now, if $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$, this means that X is a subset both from A and from B; so, $X \subset A \cap B$, and so $X \in \mathcal{P}(A \cap B)$. This proves that $\mathcal{P}(A) \cap \mathcal{P}(B) \subset \mathcal{P}(A \cap B)$.

(e) $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$; find a counterexample for the case of equality.

Answer. If $X \in \mathcal{P}(A)$, then $X \subset A$. This implies that $X \subset A \cup B$, which in turn says that $X \in \mathcal{P}(A \cup B)$. The same reasoning with the roles of A and B reversed show that $\mathcal{P}(B) \subset \mathcal{P}(A \cup B)$. Taking the two inclusions together we conclude that $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.

To see a counterexample to the equality, let $A = \{a\}$, $B = \{b\}$. Then

$$\mathcal{P}(A \cup B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\},$$

and

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}\}.$$

4. For each of the following, give a proof or counterexample. Write also the negation of each statement.

(a) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y > 5$

Answer. TRUE: given x, take for example $y = -x + 6$; then $x + y = 6 > 5$. The negation of the statement is

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y \leq 5.$$

(b) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y > 5$

Answer. FALSE: because given x, we can always choose y such that $x+y \leq 5$ (for instance, take $y = -x + 4$). The negation of the statement (which in this case is true) is

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x + y \leq 5.$$

(c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x - y > 3$

Answer. TRUE: it's very similar to (a). So, given x we can take $y = x - 4$ and then $x - y = 4 > 3$. The negation is

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x - y \leq 3.$$

(d) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x - y > 3$

Answer. FALSE: take such an x; then put $y = x - 2$, and the assertion fails. The negation (which is true) is

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, x - y \leq 3.$$

(e) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy > 2$

Answer. FALSE: take $x = 0$, then $xy = 0 < 2$ for every $y \in \mathbb{Z}$. The negation is

$$\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy \leq 2$$

(f) $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, xy > 2$

Answer. FALSE: take such an x, and then $y = 0$. Thus $xy = 0 < 2$. The negation is

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, xy \leq 2$$