Math 221-001 201710
Assignment # 3 - Answers

Read the solutions carefully, and compare them with your own work. It is important not only to check the way the assertions were proved, but also to try to learn the way of writing the proofs.

1. Show that the multiplicative identity for the integers is unique.

   Answer. Suppose that m is a multiplicative identity for $\mathbb{Z}$. Then

$$\begin{aligned} m &= 1m && \text{by (2c)} \\ &= m1 && \text{by (2d)} \\ &= 1 && \text{(since } m \text{ is a multiplicative identity).} \end{aligned}$$

2. Note that the list of postulates on the web page, which we will use, requires that $0 \neq 1$ by definition. Show that if we remove that requirement (i.e. the way the Postulates were written in the blackboard in class) then the set $\{0\}$ consisting of 0 alone, with the operations $0 + 0 = 0$, $0 \cdot 0 = 0$, satisfies all the Postulates.

   Answer. We have that $1a, 2a$ hold by hypothesis (i.e. $0 + 0 = 0$, $0 \cdot 0 = 0$). Postulates $1b$, $2b$, $1e$, $2d$, and 3 hold for any choice of integers, so they hold in particular for zero. Postulate $1c$ is satisfied from $0 + 0 = 0$; and Postulate $2c$ is satisfied by $0 \times 0 = 0$ (note that here "$1 = 0$"). The equation $0 + 0 = 0$ also makes $1d$ to hold.

   So we have to concentrate on 4 and 5. For the Order Postulates, the only possible choice is to take $\{0\}^+ = \emptyset$ (the empty set). Then $4a$ and $4b$ hold vacuously, and so does $4c$ as every element in the set $\{0\}$ is equal to zero.

   Finally, for the Induction Postulate, if $S \subset \{0\}^+ = \emptyset$, then $S = \emptyset$, so $S = \{0\}^+$. So again this postulate holds trivially.

3. Prove that if $a, b, c \in \mathbb{Z}$ and $a + b = a + c$, then $b = c$.

   Answer. We have

$$\begin{aligned} b &= b + 0 && \text{by (1c)} \\ &= b + (a + (-a)) && \text{by (1d)} \\ &= (b + a) + (-a) && \text{by (1b)} \\ &= (a + b) + (-a) && \text{by (1e)} \\ &= (a + c) + (-a) && \text{by hypothesis} \\ &= (c + a) + (-a) && \text{by (1e)} \\ &= c + (a + (-a)) && \text{by (1b)} \\ &= c + 0 && \text{by (1d)} \\ &= c && \text{by (1c).} \end{aligned}$$

4. Let $a, b \in \mathbb{Z}$. Show that

(a) If $a > 0$, $b > 0$, then $ab > 0$;

(b) if $a > 0$, $b < 0$, then $ab < 0$;

(c) if $a < 0$, $b < 0$, then $ab > 0$.

Answer.

(a) If $a > 0$, $b > 0$, then $a, b \in \mathbb{Z}^+$; so, by Postulate (4b), $ab \in \mathbb{Z}^+$, i.e. $ab > 0$.

(b) Now we have $a \in \mathbb{Z}^+$, $-b \in \mathbb{Z}^+$. So $a(-b) \in \mathbb{Z}^+$, and thus $-[a(-b)] < 0$. Then, using results from the course notes,

$$ab = -[-(ab)] = -[a(-b)] < 0.$$

(c) Here we have $-a, -b \in \mathbb{Z}^+$. So

$$ab = (-a)(-b) > 0$$

by Postulate (4b).

5. Prove that if $a, b \in \mathbb{Z}$ with $a, b \neq 0$, then $ab \neq 0$.

Answer. Since $a \neq 0$, we get from Postulate (4c) that either $a > 0$ or $a < 0$. Similarly, $b > 0$ or $b < 0$. We have four possible situations: $a > 0$ and $b > 0$; $a > 0$ and $b < 0$; $a < 0$ and $b < 0$; $a < 0$ and $b < 0$. The first three cases were contemplated in the previous question, where we proved that in those situations $ab \neq 0$ (it is either > 0 or < 0, and so cannot be zero by Postulate (4c)). The final case is $a < 0$, $b > 0$. But as $ab = ba$ (by Postulate (2d)), we can apply (ii) in question 5 with the roles of a and b interchanged. So $ab \neq 0$ also in this case.

6. Prove that if $a, b, c \in \mathbb{Z}$ with $a \neq 0$ and $ab = ac$, then $b = c$.

Answer. We want to prove that $ab = ac \implies b = c$. Suppose that $ab = ac$. Then $a(b - c) = 0$ (by appealing to Postulates (1d) and (3)). By question 5, we have $b - c = 0$ (because if $b - c \neq 0$, then question 5 implies that $a(b - c) \neq 0$). So $b = c$.