Math 221-001 201710
Assignment # 4 - Answers

1. Prove (by contradiction) that $\sqrt{1+\sqrt{2}}$ is irrational (you can use the fact that $\sqrt{2}$ is irrational; this makes the proof much simpler that the one we did for $\sqrt{2}$).

   Answer. Let us assume that $\sqrt{1+\sqrt{2}}$ is rational. That means that there exist integers m and n such that $\sqrt{1+\sqrt{2}} = m/n$. Squaring the equality we get $1+\sqrt{2} = m^2/n^2$, so $\sqrt{2} = m^2/n^2 - 1$. By writing 1 as n^2/n^2, we get

   $$\sqrt{2} = \frac{m^2 - n^2}{n^2}.$$

   This implies that $\sqrt{2}$ is rational, a contradiction. So $\sqrt{1+\sqrt{2}}$ is irrational.

2. Let k be an integer. Prove

   (a) k is odd if and only if k^2 is odd;

   (b) $k^2 + k$ is even;

   (c) if k is odd, then 8 divides $k^2 - 1$;

   (d) the product of any three consecutive integers, of which the middle one is odd, is divisible by 8.

   Answer.

   (a) If k is odd, then $k = 2p + 1$ for some integer p. Then $k^2 = (2p + 1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1 = 2r + 1$, where $r = 2p^2 + 2p$ is an integer, so k^2 is odd. To prove the reverse implication we use the contrapositive: if k is not odd, then k is even so $k = 2p$ for some integer p; then $k^2 = 4p^2 = 2(2p^2) = 2r$ where $r = 2p^2$ is an integer: thus k^2 is even, so it's not odd.

   (b) By (a), k and k^2 are both even or both odd. If they are both even, then $k = 2p$ and $k^2 = 2q$ for some integers p and q, so $k^2 + k = 2p + 2q = 2(p + q)$ even. If they are both odd, then $k = 2p + 1$, $k^2 = 2q + 1$ for some integers p and q so $k^2 + k = (2p + 1) + (2q + 1) = 2p + 2q + 2 = 2(p + q + 1)$, even.
   Another way to prove it is to note that $k^2 + k = k(k + 1)$, and that it is always true that one of k and $k + 1$ is even.

   (c) If k is odd, then $k = 2p + 1$ for some integer p. Then $k^2 - 1 = (2p + 1)^2 - 1 = 4p^2 + 4p = 4(p^2 + p)$. By part (b), $p^2 + p$ is even, so $p^2 + p = 2q$ for some integer q. So $k^2 - 1 = 4 \times 2q = 8q$, is a multiple of 8.

   (d) If we call the middle integer k, then k is odd, and the product of the three consecutive integers is $(k - 1)k(k + 1) = k(k - 1)(k + 1) = k(k^2 - 1)$. By part (c), $k^2 - 1 = 8q$ for some integer q, so $(k - 1)k(k + 1) = 8qk$, is a multiple of 8.

3. Use induction to show that $n^3 + 2n$ is a multiple of 3 for every integer $n \geq 1$. Then prove that $n^3 + 2n$ is a multiple of 3 for every $n \in \mathbb{Z}$.

   Answer. Base case: $1^3+2\times 1 = 3$ is a multiple of 3. Now assume that $k^3+2k = 3m$ for a certain m. Then

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\ &= k^3 + 2k + 3k^2 + 3k + 3 \\ &= 3m + 3k^2 + 3k + 3 \\ &= 3(m + k^2 + k + 1).\end{aligned}$$

   So $(k+1)^3 + 2(k+1)$ is a multiple of 3 (note that the expression between brackets is an integer). By induction, $n^3 + 2n$ is a multiple of 3 for all $n \geq 1$. When $n = 0$, we trivially have $0^3 + 2 \times 0 = 0 = 3 \times 0$. And when $n < 0$, we have that $-n > 0$, so

$$n^3 + 2n = -[(-n)^3 + 2(-n)].$$

   If we write $m = -n$, then $m > 0$ and $n^3 + 2n = -(m^3 + 2m)$. By the first part, since $m > 0$, there exists $k \in \mathbb{Z}$ such that $m^3 + 2m = 3k$. Then

$$n^3 + 2n = -(m^3 + 2m) = -3k = 3(-k),$$

   so $n^3 + 2n$ is a multiple of 3. We have thus shown that $n^3 + 2n$ is a multiple of 3 for all $n \in \mathbb{Z}$.

4. Prove that for all integers $n \geq 4$, $n^2 \leq 2^n$.

   Answer. We proceed by Induction. For the base case, we have $4^2 = 16$, $2^4 = 16$, so $4^2 \leq 2^4$. Now assume as inductive hypothesis that $n^2 \leq 2^n$. Then

$$(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 \leq 2^n + 2^n = 2 \times 2^n = 2^{n+1}$$

   (in the second inequality, we used the fact that $2n+1 \leq 2^n$ for all $n \geq 3$, as proved in class). So, assuming that the inequality holds for n, we have proven that it holds for $n+1$. By the Induction Principle, $n^2 \leq 2^n$ for all $n \geq 4$.

5. Prove that for all integers $n \geq 1$, $n^3 - n$ is a multiple of 6.

6. Prove that for all integers $n \geq 1$, $n^3 - n$ is a multiple of 6.

   Answer. We proceed by Induction. For $n = 1$, we have $1^3 - 1 = 0 = 6 \times 0$, a multiple of 6. Now assume as inductive hypothesis that $n^3 - n = 6h$ for an integer h. Then

$$\begin{aligned}(n+1)^3 - (n+1) &= n^3 + 3n^2 + 3n + 1 - n - 1 = n^3 - n + 3(n^2 + n) \\ &= 6h + 3(n^2 + n) = 6h + 3n(n+1).\end{aligned}$$

   By question 2, $n(n+1) = n^2 + n$ is even, i.e. $n(n+1) = 2\ell$ for an integer ℓ. So

$$(n+1)^3 - (n+1) = 6h + 3n(n+1) = 6h + 3 \times 2\ell = 6h + 6\ell = 6(h+\ell).$$

   So $(n+1)^3 - (n+1)$ is a multiple of 6 provided that $n^3 - n$ is, and so $n^3 - n$ is a multiple of 6 for all positive integers (it is actually true for all integers as can be proven mimicking the argument in question 3.

7. Prove that, for every integer $n \geq 1$,

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

Answer. We proceed by Induction. For the base case, let $n = 1$; then the two sides in our equality are

$$\sum_{j=1}^{1} j^2 = 1^2 = 1, \qquad \text{and} \qquad \frac{1(1+1)(2\times 1+1)}{6} = \frac{2\times 3}{6} = 1. \tag{1}$$

Now that assume that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ for some n. Then

$$\begin{aligned}
\sum_{j=1}^{n+1} j^2 &= \sum_{j=1}^{n} j^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= (n+1)\left[\frac{n(2n+1)}{6} + n + 1\right] = (n+1)\left[\frac{n(2n+1)+6n+6}{6}\right] \\
&= (n+1)\left[\frac{n(2n+3)+4n+6}{6}\right] = (n+1)\left[\frac{n(2n+3)+2(2n+3)}{6}\right] \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.
\end{aligned}$$

So the formula holds for $n+1$ too. Then, by the Induction Principle, (1) holds for all positive integers n.