Math 221-001 201710

Assignment # 5 – Answers

1. Let $a, b, c, d \in \mathbb{Z}$. Prove

   (a) if $a|b$ and $c|d$, then $ac|bd$;

   Answer. We have that $b = ka$ and $d = ch$ for some integers k and h. Then $bd = (ka)(ch) = (kh)(ac)$; as kh is an integer, we have that $ac|bd$.

   (b) if $ab|c$, then $a|c$;

   Answer. We have that $c = k(ab)$ for some integer k; then $c = (kb)a$, and since kb is an integer we conclude that $a|c$.

   (c) if $a|b$ and $b|c$ then $a|c$;

   Answer. We have that $a|b$ and $b|c$, that is $b = ka$ and $c = hb$ for certain integers k and h. Then $c = hb = h(ka) = (hk)a$; as hk is an integer, we have that $a|c$.

   (d) if $a|b$ and $a|c$ then $a|(b + c)$.

   Answer. We have that $b = ka$ and $c = ha$ for some integers k and h. Then $b + c = ka + ha = (k + h)a$; as $k + h$ is an integer, we have that $a|(b + c)$.

2. For each a, b find the unique $q, r \in \mathbb{Z}$ such that $a = qb + r$, $0 \leq r < b$.

   (a) $a = 256$, $b = 4$;

   (b) $a = -256$, $b = 4$;

   (c) $a = 255$, $b = 4$;

   (d) $a = -255$, $b = 4$.

   Answer.

   (a) $256 = 2^8 = 4^4$, so $256 = 64 \times 4$, that is $q = 64$, $r = 0$.

   (b) $-256 = (-64) \times 4$, so $q = -64$, $r = 0$.

   (c) $255 = 256 - 1 = 64 \times 4 - 1 = 63 \times 4 + 3$, so $q = 63$, $r = 3$.

   (d) $-255 = -63 \times 4 - 3 = -64 \times 4 + 4 - 3 = -64 \times 4 + 1$, so $q = -64$, $r = 1$.

3. Find the greatest common divisor of 252 and -180, and also integers m and n such that it can be written as $252m - 180n$.

   Answer. Because greatest common divisors are positive, we can just find $\gcd(252, 180)$. We use the Euclidean Algorithm:

$$252 = 1 \times 180 + 72;$$
$$180 = 2 \times 72 + 36;$$
$$72 = 2 \times 36 + 0.$$

So $\gcd(180, 252) = 36$. To express it as a linear combination, we go backwards through the previous equations:

$$\begin{aligned} 36 &= 180 - 2 \times 72 = 180 - 2 \times (252 - 180) \\ &= 3 \times 180 - 2 \times 252 \\ &= 252 \times (-2) + (-180) \times (-3) \\ &= 252 \times (-2) - 180 \times (-3). \end{aligned}$$

4. Find the smallest integer in the set

$$\{x \in \mathbb{Z}^+ : x = 6s + 15t \text{ for some } s, t \in \mathbb{Z}\},$$

and also integers m and n such that it can be written as $6m + 15n$.

Answer. The smallest integer in the set is $gcd(6, 15)$, which equals 3:

$$\begin{aligned} &15 = 2 \times 6 + 3; \\ &6 = 2 \times 3 + 0. \end{aligned}$$

So $\gcd(6, 15) = 3$. Since $3 = 15 - 2 \times 6$, $n = 1$, $m = -2$.

5. Find the smallest integer of the set

$$\{x \in \mathbb{Z}^+ : x = 3780p + 1200q \text{ for some } p, q \in \mathbb{Z}\},$$

and also integers m and n such that it can be written as $3780m + 1200n$.

Answer. The required smallest integer is $\gcd(3780, 1200)$, because we know that the greatest common divisor has that property. To find it, we use the Euclidean Algorithm:

$$\begin{aligned} 3780 &= 3 \times 1200 + 180 \\ 1200 &= 6 \times 180 + 120 \\ 180 &= 1 \times 120 + 60 \\ 120 &= 2 \times 60 + 0. \end{aligned}$$

So $\gcd(3780, 1200) = 60$. Going backwards through the equations,

$$\begin{aligned} 60 &= 180 - 120 = 180 - (1200 - 6 \times 180) = 7 \times 180 - 1200 \\ &= 7(3780 - 3 \times 1200) - 1200 = 7 \times 3780 + (-22) \times 1200. \end{aligned}$$

So $m = 7$, $n = -22$.

This can also be done with the shorter method, as follows:

1	0	3780
0	1	1200
1	−3	180
−6	19	120
7	−22	60

(it is not really that much shorter, but it is a lot better at keeping track of the numbers that otherwise might be lost with so many brackets to cancel).

6. Let $a, b, m, n \in \mathbb{Z}$ such that $ma + nb = 1$. Show that $\gcd(a, b) = 1$.

   Answer. We know that $\gcd(a, b)$ is the least integer than can be written as $ma+nb$. Since 1 can be written in that form and 1 is the least positive integer, 1 has to be the least integer that can be written in that form, so $1 = \gcd(a, b)$.

7. Find the general solution of the following Diophantine equations:

   (a) $212x + 37y = 1$;
   Answer. For a solution to exist, we need $\gcd(212, 37) = 1$; this certainly happens, since 37 is prime. We do the Euclidean Algorithm:

1	0	212
0	1	37
1	−5	27
−1	6	10
3	−17	7
−4	23	3
11	−63	1

   So $x = 11$, $y = -63$ is a solution. Thus the general solution is given by

$$x = 11 - 37n, \quad y = -63 + 212n.$$

   (b) $91x + 126y = 203$;
   Answer. The GCD between 91 ($= 7 \times 13$) and 126 ($= 2 \times 7 \times 9$) is 7. For a solution to exist we need 203 to be a multiple of 7, which it is: $203 = 7 \times 29$. We do the Euclidean Algorithm:

1	0	126
0	1	91
1	−1	35
−2	3	21
3	−4	14
−5	7	7
13	−18	0

   So $y = 29 \times (-5) = -145$, $x = 29 \times 7 = 203$ is a solution. The general solution is given by

$$x = 203 - 18n, \quad y = -145 + 13n.$$

   (c) $169x - 65y = 91$.
   Answer. $\gcd(169, 65) = 13$. Since $13|91$, there is solution. We do the Euclidean Algorithm:

1	0	169
0	1	65
1	−2	39
−1	3	26
2	−5	13
−5	13	0

Recall that $91 = 13 \times 7$. So $x = 7 \times 2 = 14$, $y = -7 \times (-5) = 35$ is a solution (the negative sign comes from the fact that there is a negative sign in front of 65: otherwise we would be solving $169x + 65y = 91$). The general solution is given by

$$x = 14 + 5n, \quad y = 35 + 13n.$$

8. Can 120 be expressed as the sum of two positive integers, one of which is divisible by 11 and the other by 17?

   Answer. We want to see whether it is possible to write $120 = 11x + 17y$, with both x, y positive. Since both 11 and 17 are prime, $\gcd(11, 17) = 1$. So there exist integers m, n such that $1 = 11m + 17n$. Then,

$$120 = 11(120m) + 17(120n).$$

   To find m and n we can use the Euclidean Algorithm:

1	0	17
0	1	11
1	−1	6
−1	2	5
2	−3	1

   So, $1 = 17 \times 2 + 11 \times (-3)$. Multiplying by 120, we get

$$120 = 17 \times 240 + 11 \times (-360).$$

   We can even express the general form of such m and n (note that $GCD(360, 240) = 120$):

$$x = 240 - 11k, \quad y = -360 + 17k, \quad \text{with } k \text{ an integer.}$$

   Finally, we need to ask ourselves whether some of those solutions can be both positive. We are looking for two positive integers that add to 120: this means that we want solutions where x and y are both positive.The values of x greater than 0 are those where $240 - 11k > 0$, i.e. $k < 240/11 = 21.8$. So, $k \leq 21$.

   We also need $y \geq 0$, that is $-360 + 17k > 0$. This implies $k > 360/17 = 21.1$, that is $k \geq 22$. So there is no way to make both x, y positive at the same time.