Math 221-001 201710
Assignment # 8 – Answers

1. Show that inclusion is an order relation in the set of all sets.

   Answer. We need to check that inclusion is reflexive, antisymmetric, and transitive.

   Reflexivity is the fact that, for any set A, $A \subset A$.

   If $A \subset B$ and $B \subset A$, then $A = B$ (this is actually the definition of equality of sets!), so $\subset$ is antisymmetric.

   For transitivity, if $A \subset B \subset C$, then every element of A is an element of B, and every element of B is an element of C. So every element of A is an element of C, i.e. $A \subset C$.

2. For the following sets and relations, determine whether they are equivalence relations. If they are, determine all equivalence classes.

   (a) $X = \{1, 2, 3\}$, and $R = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.
   Answer. This relation is not reflexive, because $3R3$ is false (because $(3, 3) \notin R$). So it is not an equivalence relation.

   (b) $X = \{1, 2, 3\}$, and $S = \{(1, 1), (2, 2), (3, 3)\}$.
   Answer. S is reflexive, because for every $x \in X$, $(x, x) \in R$. It is symmetric, because if $(x, y) \in R$, this implies that $y = x$, so $(y, x) \in R$. It is transitive, because if xRy and yRz, we have that $x = y$ (from the first relation) and $y = z$ (from the second relation). So $x = z$ and thus xRz.
   Every element is related only to itself, so the equivalence class of each element consists of the element alone. So,

   $$[1] = \{1\}, \quad [2] = \{2\}, \quad [3] = \{3\}.$$

   (This can be made a lot shorter by realizing that S is equality)

   (c) $\mathbb{Z}$, and $R = \{(n, m) : n = m + 1\}$.
   Answer. This relation is not reflexive, since the equation $n = n + 1$ has no solution for any integer. So it is not an equivalence relation.

   (d) $\mathbb{Z}$, and $R = \{(n, m) : 3|n - m\}$.
   Answer. This should be already well-known, but it doesn't hurt doing it again. The relation is reflexive: for any integer n, $n - n = 0 = 3 \times 0$, so nRn. It is symmetric: if $n - m = 3k$ for some $k \in \mathbb{Z}$, then $m - n = -(n - m) = -3k = 3(-k)$. And it is transitive, because if $n - m = 3k$, $m - p = 3l$, with $k, l \in \mathbb{Z}$, then $n - p = 3k - m + (m - 3l) = 3k + 3l = 3(k + l)$. There are three equivalence classes:

   $$[0] = \{3k : k \in \mathbb{Z}\}, \quad [1] = \{3k + 1 : k \in \mathbb{Z}\}, \quad [2] = \{3k + 2 : k \in \mathbb{Z}\}.$$

3. List all possible equivalence relations on the set $\{a, b, c\}$.

   Answer. Each equivalence relation will correspond to a partition of the set. The partitions (and its corresponding equivalence relation)are

   (a) $\{a\}, \{b\}, \{c\}$. The equivalence relation is equality:

   $$R_1 = \{(a, a), (b, b), (c, c)\}.$$

   (b) $\{a, b\}, \{c\}$. The equivalence relation is

   $$R_2 = \{(a, a), (b, b), (a, b), (b, a), (c, c)\}.$$

   (c) $\{a\}, \{b, c\}$. The equivalence relation is

   $$R_3 = \{\{(a, a), (b, b), (c, c), (b, c), (c, b)\}.$$

   (d) $\{a, c\}, \{b\}$. The equivalence relation is

   $$R_4 = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}.$$

   (e) $\{a, b, c\}$. The equivalence relation is "all related to all",

   $$R_5 = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, b), (c, c), (a, c), (c, a)\}.$$

4. Find a relation R, on a set S, that is symmetric and transitive, but not reflexive.

   Answer. The first example in the previous question would do. But we can find an even simpler example. The trick is to use the same idea: let $S = \{1, 2\}$, and $R = \{(1, 1)\}$. Then the relation is symmetric, because if aRb, both $a = b = 1$, so bRa; and it is transitive, because if aRb and bRc, then $a = b = c = 1$, so aRc. But it is not reflexive, because it is not true that $2R2$.

5. The following argument shows that if a relation is symmetric and transitive, then it is reflexive. If you did question 4 right, then you know this has to be false. Find the error in the argument

   > "Assume that R is symmetric and transitive on S. For any $a, b \in S$, aRb implies bRa, because R is symmetric. Then, from aRb and bRa we conclude by transitivity aRa. Since aRa, R must be reflexive."

   Answer. For the relation to be reflexive, we need this argument to work for any a. But it could happen – as in the example in 4 – that a is not related to **any** element in the set. Then the first step aRb cannot be done, and the reasoning is incorrect.