Math 221-001 201710
Assignment # 9

Due: April 7th.

1. Below we write H for the set of all human beings. Decide if each of the following relations is a function.

   (a) $A = \{(t, s) : \ s^2 = t\} \subset \mathbb{R} \times \mathbb{R}$;

   Answer. No. For instance, A contains the pairs $(1, 1)$ and $(1, -1)$, which shows that the relation does not define a function on 1 (and the argument can be used with any other point).

   (b) $B = \{(t, s) : \ t^2 = s\} \subset \mathbb{R} \times \mathbb{R}$;

   Answer. Yes. Given a pair (t, a), then necessarily $a = t^2$. So B is the function $t \mapsto t^2$.

   (c) $C = \{(x, y) : \ x \text{ is the mother of } y\} \subset H \times H$;

   Answer. No. Given $(x, y) \in C$, we have that x is the mother of y. So, for a mother x with two daughters y, z, the pairs (x, y) and (x, z) belong to C. But there is something deeper here: a function should be defined on every element of its domain. Not every human being is a mother, so this cannot be a function.

   (d) $D = \{(x, y) : \ x \text{ is the daughter of } y\} \subset H \times H$;

   Answer. No. Given any x, y could be either her mother or her father. As the choice is not unique, we do not have a function. As in the previous question, here not every human is a "daughter"; only females are.

   (e) $E = \{(a, b) : \ a \text{ is the age of } b\} \subset \mathbb{Z} \times H$;

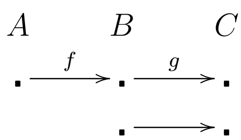
   Answer. No. Many people have the same age. So, if Alice and Bob are 30 years old, then the pairs $(30, \text{Alice})$, $(30, \text{Bob})$ are in E. And again the same thing that happened in the previous questions arises: not every integer is the age of a human being: the relation is not defined for every integer, and so it cannot be a function.

   (f) $F = \{(a, b) : \ b \text{ is the age of } a\} \subset H \times \mathbb{Z}$.

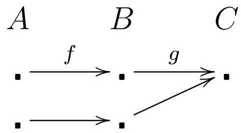
   Answer. Yes. Given a person a, then b will his/her age.

2. Let $f : A \to B$ and $g : B \to C$, such that f is injective and g is surjective. Is $g \circ f$ surjective? Is it injective? Provide proof/counterexample.

   Answer. First an example with f is injective, g is surjective, but $g \circ f$ is not surjective:

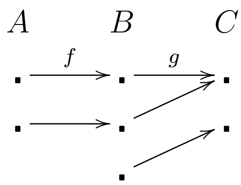


Now an example where f is injective, g is surjective, but $g \circ f$ is not injective:



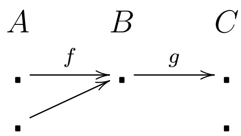
(1)

We can combine the ideas of both examples to get an example where f is injective, g is surjective, but $g \circ f$ is neither surjective nor injective:



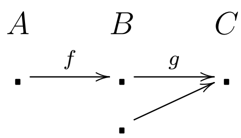
3. Let $f : A \to B$ and $g : B \to C$, such that f is surjective and g is injective. Is $g \circ f$ surjective? Is it injective? Provide proof/counterexample.

   Answer. Here is an example where f is surjective, g is injective, but $g \circ f$ is neither surjective nor injective:



4. Find an example of functions $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is surjective but f is not surjective.

   Answer. The third example in question 2 fits the bill. To make an example where sets have two or less elements:



(2)

5. Let $f : A \to B$. Prove the following:

   (a) If f is invertible, then f is bijective;

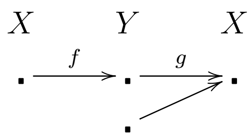
   Answer. Let $g = f^{-1}$. If $f(x) = f(y)$, then $x = g(f(x)) = g(f(y)) = y$, so $x = y$; thus f is injective. Given any $y \in B$, we have $y = f(g(y))$, so f is surjective.

   (b) If f is bijective, then f is invertible.

   Answer. Define $g : B \to A$ in the following way. For any $b \in B$, since f is onto there exists $a_b \in A$ with $f(a_b) = b$; such an a is unique because f is injective. So we put $g(b) = a_b$. Then $f(g(b)) = f(a_b) = b$. And, for any $a \in A$, $g(f(a)) = a_{f(a)}$; but $a_{f(a)}$ is the unique element in A with $f(a_{f(a)}) = f(a)$; as f is injective, $a_{f(a)} = a$, so $g(f(a)) = a$. This g in the inverse of f.

6. The identity function $\mathrm{id}_X : X \to X$ is sometimes denoted by $1_X : X \to X$ (that is, $1_X(x) = x$ for every $x \in X$). Find an example of sets X, Y and functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = 1_X$ but $f \circ g \neq 1_Y$.

   Answer. From $g \circ f = 1_X$ we deduce that f has to be injective. But we want it not to be invertible, so it cannot be surjective. This suggests the following:

   (3)

   If we denote $X = \{1\}$, $Y = \{a, b\}$, we have $g(f(1)) = g(a) = 1$, so $g \circ f = 1_X$. But $f(g(b)) = f(1) = a \neq b$, so $f \circ g \neq 1_Y$.

7. Consider, for each pair of integers m and n, the function $f_{m,n} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f_{m,n}(x, y) = mx + ny$ (that is, different choices of m and n give you different functions). Decide which values of m and n (if any) make $f_{m,n}$ surjective, and which values of m and n (if any) make $f_{m,n}$ injective.

   Answer. The function is never injective: if both $m = n = 0$, then $f(x, y) = 0$ and it is not injective. If $m \neq 0$, then $f(-n, m) = f(0, 0)$ and $(-n, m) \neq (0, 0)$. For f to be surjective, we should be able to write any integer z as $mx + ny$. So, if f is surjective, we would have $1 = mx + ny$ for certain $x, y \in \mathbb{Z}$, and so m and n are coprime. Conversely, if m and n are coprime then the equation $z = mx + ny$ has a solution for every z and thus f would be surjective. So, f is surjective if and only if m and n are coprime.