Math 221-001 201710
Assignment # 10 – Answers

1. Prove that the relation between sets given by $|A| = |B|$ is an equivalence relation.

   Answer. The identity function $1_A : A \to A$ is a bijection, so $|A| = |A|$; thus, the relation is reflexive. If $|A| = |B|$, there exists a bijection $f : A \to B$.

   We can then consider its inverse $f^{-1} : B \to A$, which is also a bijection, so $|B| = |A|$, and the relation is symmetric.

   If $|A| = |B|$ and $|B| = |C|$, we have bijections $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ is a bijection and $|A| = |C|$. So the relation is transitive.

2. Let $A \subset \mathbb{N}$ be an infinite set. Show that A is countable.

   Answer. We need to construct a bijection $f : \mathbb{N} \to A$. We know that every set of positive integers has a least element (we proved that in class a very long time ago, using induction). So let a_1 be the least element of A. Now let a_2 be the least element of $A \setminus \{a_1\}$, a_3 the least element of $A \setminus \{a_1, a_2\}$, and in general a_{k+1} the least element of $A \setminus \{a_1, \ldots, a_k\}$.

   Define $f(k) = a_k$. By construction, each a_{k+1} is different from all its predecessors, so f is injective. To see that f is surjective, suppose it isn't; then there exists $a \in A$ such that $a \neq A_k$ for all $k \in \mathbb{N}$. Because of the choice of each a_k, we have that $a \geq a_k$ for all k. It is easy to show by induction that $a_k \geq k$ for all $k \in \mathbb{N}$, and so we get $a \geq k$ for all $k \in \mathbb{N}$. In particular $a \geq a + 1$, a contradiction. So f is surjective.

3. Write a proof that $|\mathbb{R}| \neq |\mathbb{N}|$.

   Answer. We prove that there is no surjection $f : \mathbb{N} \to \mathbb{R}$. Let $f : \mathbb{N} \to \mathbb{R}$ be any function. For each n, $f(n)$ is a real number. Construct a real number c in the following way: the n^{th} decimal of c is

   - 6 if the n^{th} decimal of $f(n)$ is not 6;
   - 7 if the n^{th} decimal of $f(n)$ is 6.

   The number c constructed like that is different from $f(n)$, for each n, because they differ on the n^{th} decimal digit. So c is not in the image of f, and f is not surjective.

4. Show that $|\mathbb{Q}| \neq |\mathbb{R}|$.

   Answer. We know that $|\mathbb{Q}| = |\mathbb{N}|$, and that $|\mathbb{N}| \neq |\mathbb{R}|$. If we had a bijection $\alpha : \mathbb{Q} \to \mathbb{R}$, we could compose it with a bijection $\beta : \mathbb{N} \to \mathbb{Q}$ to get a bijection $\beta \circ \alpha : \mathbb{N} \to \mathbb{R}$, which we know is impossible by the previous answer.