University of Regina
Department of Mathematics and Statistics
Math 221-001
Midterm - Answers

[5 Marks] 1. Prove, using the Postulates, that for any integer n, $-n = (-1)n$. If you need to you can use, without proving it, the fact that $0 \cdot n = 0$.

Answer. We have

$$\begin{aligned} n + (-1)n &= 1n + (-1)n \quad \text{(using 2c)} \\ &= (1 + (-1))n \quad \text{(using 3)} \\ &= 0n \quad \text{(using 1d)} \\ &= 0. \quad \text{(using the information from the question statement)} \end{aligned}$$

If we now add $-n$ to both sides of the equality above, we get

$$\begin{aligned} -n &= -n + 0 \quad \text{(1c)} \\ -n &= -n + (n + (-1)n) \quad \text{(proven above)} \\ &= (-n + n) + (-1)n) \quad \text{(1b)} \\ &= (n + (-n)) + (-1)n) \quad \text{(1e)} \\ &= 0 + (-1)n) \quad \text{(1d)} \\ &= (-1)n + 0 \quad \text{(1e)} \\ &= (-1)n. \quad \text{(1e)} \end{aligned}$$

Looking at the last string of equalities, we have shown that $-n = (-1)n$.

One can make this idea into a slightly shorter proof as follows:

$$\begin{aligned} -n &= -n + 0 \quad \text{(1c)} \\ &= -n + 0n \quad \text{(information from the question)} \\ &= -n + (1 + (-1))n \quad \text{(1d)} \\ &= -n + ((n + (-1)n) \quad \text{(3)} \\ &= (-n + n) + (-1)n \quad \text{(1b)} \\ &= (n + (-n)) + (-1)n \quad \text{(1e)} \\ &= 0 + (-1)n \quad \text{(1d)} \\ &= (-1)n + 0 \quad \text{(1e)} \\ &= (-1)n \quad \text{(1c)} \end{aligned}$$

Comments. *An alarmingly common mistake was to start the proof with* $-n = (-1)n$*. That's the assertion you wanted to prove, so it should be your* *conclusion**, not your* *hypothesis**! Many students use the postulates more or less correctly to get from* $-n = (-1)n$ *to to* $0 = 0$*. But you didn't want to prove that* $0 = 0$*, you wanted to prove that* $-n = (-1)n$*. In some of those cases, the steps were reversible so writing them from bottom to top would have made a correct proof.*

Postulates for the Integers

1. **Addition Postulates**
    a. If $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$.
    b. If $a, b, c \in \mathbb{Z}$, then $(a + b) + c = a + (b + c)$ (associativity).
    c. There exists $0 \in \mathbb{Z}$ such that $a + 0 = 0$ for all $a \in \mathbb{Z}$ (identity element).
    d. For each $a \in \mathbb{Z}$ there exists $-a \in \mathbb{Z}$ such that $a + (-a) = 0$ (additive inverse).
    e. For all $a, b \in \mathbb{Z}$, $a + b = b + a$.
2. **Multiplication Postulates**
    a. If $a, b \in \mathbb{Z}$, then $ab \in \mathbb{Z}$.
    b. If $a, b, c \in \mathbb{Z}$, then $(ab)c = a(bc)$ (associativity).
    c. There exists $1 \in \mathbb{Z}$ such that $1 \neq 0$ and $1a = a$ for all $a \in \mathbb{Z}$ (identity element).
    d. For all $a, b \in \mathbb{Z}$, $ab = ba$.
3. **Distributive Law** For all $a, b, c \in \mathbb{Z}$,

$$a(b + c) = ab + ac.$$

4. **Order Axioms.** The is a subset $\mathbb{Z}^+ \subset \mathbb{Z}$ such that
    a. For all $a, b \in \mathbb{Z}^+$, $a + b \in \mathbb{Z}^+$.
    b. For all $a, b \in \mathbb{Z}^+$, $ab \in \mathbb{Z}^+$.
    c. (Trichotomy Law) For each $a \in \mathbb{Z}$, exactly one of the following holds:
        i. $a \in \mathbb{Z}^+$
        ii. $a = 0$
        iii. $-a \in \mathbb{Z}^+$
    5. **Induction.** If $S \subset \mathbb{Z}^+$ and
        a. $1 \in S$, and
        b. $a \in S$ always implies $a + 1 \in S$,

        then $S = \mathbb{Z}^+$.

[10 Marks] 2. Show that $P \vee Q \vee R \equiv (\sim P \wedge \sim Q) \implies R$.

With a truth table:

P	Q	R	$P \vee Q \vee R$	$\sim P \wedge \sim Q$	$(\sim P \wedge \sim Q) \implies R$
T	T	T	T	F	T
F	T	T	T	F	T
T	F	T	T	F	T
F	F	T	T	T	T
T	T	F	T	F	T
F	T	F	T	F	T
T	F	F	T	F	T
F	F	F	F	T	F

A way to do this is without the truth table is to note that $P \vee Q \vee R$ will be true unless all three P, Q, R are false. For the implication $(\sim P \wedge \sim Q) \implies R$ to be false, we need R to be false, and $\sim P \wedge \sim Q \equiv \sim (P \vee Q)$ to be true; thus we need $P \vee Q$ to be false, which only happens when P, Q are both false; in summary, the implication can only be false when all three P, Q, R are false, exactly as $P \vee Q \vee R$.

[10 Marks] 3. For each statement, give a proof or a counterexample. Write also the negation.

(a) $\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z},\ a - b > 5$

Answer. If such a exists, we would have $a > b + 5$ for all b; in particular, we would get $a > a + 5$, which is impossible. In other words, given a, take $b = a$ and the statement says that $0 > 5$, which is not true. So the statement is false. The negation is

$$\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, a - b \leq 5.$$

(b) $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}, a - b > 5$.

Answer. Given a, we can take $b = a - 6$. Then $a - b = a - (a - 6) = 6 > 5$. So the statement is true. The negation is

$$\exists a \in \mathbb{Z}, \forall b \in \mathbb{Z}, a - b \leq 5.$$

Comments. *I saw lots of confusion with the logic here. In part (a), many were using "an example" to show that the statement was false; that doesn't cut it. Consider the following existential statement: "there exists a Canadian who went to Mars". The statement is false, but you cannot say that it is false because Justin Trudeau did not go to Mars; you have to show that not a single Canadian has been there.*

[10 Marks] 4. Prove by induction that, for every $n \in \mathbb{Z}^+$, the number $5^n - 1$ is a multiple of 4.

Answer. For $n = 1$, we have $5^1 - 1 = 4$ is a multiple of 4. Now assume that $5^k - 1 = 4h$ for some h. Then

$$5^{k+1} - 1 = 5 \times 5^k - 1 = 5(4h + 1) - 1 = 20h + 4 = 4(5h + 1).$$

As $5h + 1 \in \mathbb{Z}$, we get that 5^{k+1} is a multiple of 4. By induction, then $5^n - 1$ is a multiple of 4 for all $n \in \mathbb{Z}^+$.