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University of Regina
Department of Mathematics and Statistics
Math 221-001
Midterm 2 - Answers

[7 Marks] 1. Prove that two consecutive integers are coprime.

Answer. For $n \in \mathbb{Z}$, we have $(n+1) - n = 1$. As $\gcd(n, n+1)$ is the least integer that can be written in the form $a(n+1) + bn$, we get that $1 = \gcd(n, n+1)$.

Second proof. Let $d = \gcd(n, n+1)$. Then $n = kd$, $n+1 = hd$ for some $k, h \in \mathbb{Z}$. Subtracting the two equalities, we get $1 = (h-k)d$. As the only positive divisor of 1 is 1 (proven in class), $d = 1$.

Third proof. Using the Euclidean Algorithm: since $n+1 = 1 \times n + 1$, we have

$$\gcd(n+1, n) = \gcd(n, 1) = 1.$$

[8 Marks] 2. Solve $3x \equiv 2 \pmod{11}$.

Answer. Since 11 is prime, we know that 3 is invertible modulo 11. We have $3 \times 4 = 12 \equiv 1 \pmod{11}$. So, multiplying our equation by 4, we get

$$x \equiv 8 \pmod{11}.$$

The solution is then $x = 8 + 11n$, $n \in \mathbb{Z}$.

[10 Marks] 3. Solve each Diophantine equation below. Answer also whether there is a positive solution.

(a) $460x + 805y = 231$

We look first for $\gcd(460, 805)$. Using the Euclidean Algorithm,

1	0	805	
0	1	460	1
1	−1	345	1
−1	2	115	3
		0	

So $\gcd(805, 460) = 115$. As 231 is not a multiple of 115, the equation has no solution.

(b) $460x + 805y = 230$

Answer. We already know that $\gcd(805, 460) = 115$. From $460 = 4 \times 115$, $805 = 7 \times 115$ and $230 = 2 \times 115$, we reduce our equation to

$$4x + 7y = 2.$$

We see that $4 \times 4 + 7 \times (-2) = 2$ (or, if this is not obvious enough, we first do $4 \times 2 + 7 \times (-1) = 1$, and then multiply by 2). This gives us the solutions

$$x = 4 + 7n, \quad y = -2 - 4n, \quad n \in \mathbb{Z}.$$

[10 Marks]

4. Find the last two digits of 9^{4325}.

Answer. We have $9^2 = 81$,

$$9^4 = 81^2 \equiv 61 \pmod{100}, \quad 9^8 \equiv 61^2 \equiv 21 \pmod{100}, \quad 9^{16} \equiv 21^2 \equiv 41 \pmod{100}.$$

Then

$$9^{32} \equiv 41^2 \equiv 81 \pmod{100}.$$

This means that $9^{32} \equiv 9^2 \pmod{100}$. As $(9, 100) = 1$, we know that 9 is invertible module 100, and so we get $9^{30} \equiv 1 \pmod{100}$. As $4325 = 30q + 5$ for some q (explicitly, $q = 144$),

$$9^{4325} = (9^{30})^{144} 9^5 \equiv 9^5 = 9^4 \times 9 \equiv 61 \times 9 \equiv 49 \pmod{100}.$$

A similar, but maybe easier approach: Since $61 \times 21 \equiv 81 \pmod{100}$, we get that $9^4 \times 9^8 \equiv 9^2 \pmod{100}$. So we get $9^{10} \equiv 1 \pmod{100}$, and we can proceed as above.