Math 122-001 201730
Practice Assignment # 1 – Answers

1. For each linear system, solve by doing elementary row operations. If there are infinitely many solutions, express them in parametric form **in two different ways.**

(a) $\begin{cases} 3x_1 - x_2 = 3 \\ 4x_1 + x_2 = 1 \end{cases}$

Answer. We work on the augmented matrix of the system.

$$\left[\begin{array}{rr|r} 3 & -1 & 3 \\ 4 & 1 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{rr|r} 3 & -1 & 3 \\ 1 & 2 & -2 \end{array}\right] \xrightarrow{R1-3R2} \left[\begin{array}{rr|r} 0 & -7 & 9 \\ 1 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{R2+\frac{2}{7}R1} \left[\begin{array}{rr|r} 0 & -7 & 9 \\ 1 & 0 & 4/7 \end{array}\right] \xrightarrow{-\frac{1}{7}R1} \left[\begin{array}{rr|r} 0 & 1 & -9 \\ 1 & 0 & 4/7 \end{array}\right]$$

So there is a unique solution, $x_1 = 4/7$, $x_2 = -9/7$.

(b) $\begin{cases} 3x - 6y + 5z = 2 \\ 4x - 3y + z = 6 \end{cases}$

Answer. We work on the augmented matrix of the system.

$$\left[\begin{array}{rrr|r} 3 & -6 & 5 & 2 \\ 4 & -3 & 1 & 6 \end{array}\right] \xrightarrow{R2\leftrightarrow R1} \left[\begin{array}{rrr|r} 4 & -3 & 1 & 6 \\ 3 & -6 & 5 & 2 \end{array}\right] \xrightarrow{R1-R2} \left[\begin{array}{rrr|r} 1 & 3 & -4 & 4 \\ 3 & -6 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R2-3R1} \left[\begin{array}{rrr|r} 1 & 3 & -4 & 4 \\ 0 & -15 & 17 & -10 \end{array}\right] \xrightarrow{R1+\frac{1}{5}R2} \left[\begin{array}{rrr|r} 1 & 0 & -3/5 & 2 \\ 0 & -15 & 17 & -10 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{15}R2} \left[\begin{array}{rrr|r} 1 & 0 & -3/5 & 2 \\ 0 & 1 & -17/15 & 2/3 \end{array}\right]$$

By choosing z as a parameter t, we get the infinitely many solutions in the form

$$x = 2 + \frac{3t}{5}, \quad y = \frac{2}{3} + \frac{17t}{15}, \quad z = t.$$

We can get a different parametrization if for example we work with the column for z first, and then y. We have

$$\left[\begin{array}{rrr|r} 3 & -6 & 5 & 2 \\ 4 & -3 & 1 & 6 \end{array}\right] \xrightarrow{R1-5R2} \left[\begin{array}{rrr|r} -17 & 9 & 0 & -28 \\ 4 & -3 & 1 & 6 \end{array}\right] \xrightarrow{R2+\frac{1}{3}R1} \left[\begin{array}{rrr|r} -17 & 9 & 0 & -28 \\ -5/3 & 0 & 1 & -10/3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}R1} \left[\begin{array}{rrr|r} -17/9 & 1 & 0 & -28/9 \\ -5/3 & 0 & 1 & -10/3 \end{array}\right]$$

Taking x as a parameter t, we get the solutions expressed as

$$x = t, \quad y = \frac{17x}{9} - \frac{28}{9}, \quad z = \frac{5x}{3} - \frac{10}{3}.$$

(c) $\begin{cases} 4x + 7y = 1 \\ 5x + 8y = 2 \\ 6x + 9y = 3 \end{cases}$

Answer. We work on the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 4 & 7 & 1 \\ 5 & 8 & 2 \\ 6 & 9 & 3 \end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{cc|c} 4 & 7 & 1 \\ 5 & 8 & 2 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{cc|c} 4 & 7 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{cc|c} 4 & 7 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R1-4R2} \left[\begin{array}{cc|c} 0 & 3 & -3 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R1} \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{cc|c} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

So the only solution is $x = 2$, $y = -1$.

(d) $\begin{cases} 4x + 7y = 1 \\ 5x + 8y = 2 \\ 6x + 9y = 4 \end{cases}$

Answer. We work on the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 4 & 7 & 1 \\ 5 & 8 & 2 \\ 6 & 9 & 4 \end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{cc|c} 4 & 7 & 1 \\ 5 & 8 & 2 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R2-R1} \left[\begin{array}{cc|c} 4 & 7 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{R3-R2} \left[\begin{array}{cc|c} 4 & 7 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

At this state we see that the third equation has become $0 = 1$, which is impossible. The system has no solution.

(e) $\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 6 \\ x_1 + 3x_2 + 5x_3 + 7x_4 + 9x_5 = 11 \\ x_1 - x_2 \phantom{{}+ 3x_3 + 4x_4 + 5x_5} = 0 \end{cases}$

Answer. We work on the augmented matrix of the system.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 7 & 9 & 11 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R2-R3]{R1-R3} \left[\begin{array}{ccccc|c} 0 & 3 & 3 & 4 & 5 & 6 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R2-R1} \left[\begin{array}{ccccc|c} 0 & 3 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R3 \leftrightarrow R1} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 3 & 3 & 4 & 5 & 6 \end{array}\right]$$

$$\xrightarrow{R3-3R2}\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -3 & -5 & -7 & -9 \end{array}\right]\xrightarrow{-\frac{1}{3}R3}\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 5/3 & 7/3 & 3 \end{array}\right]$$

$$\xrightarrow{R2-2R3}\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/3 & -2/3 & -1 \\ 0 & 0 & 1 & 5/3 & 7/3 & 3 \end{array}\right]\xrightarrow{R1+R2}\left[\begin{array}{ccccc|c} 1 & 0 & 0 & -1/3 & -2/3 & -1 \\ 0 & 1 & 0 & -1/3 & -2/3 & -1 \\ 0 & 0 & 1 & 5/3 & 7/3 & 3 \end{array}\right]$$

At this stage all rows have a leading one and we are in the reduced row echelon form. In this form, we want to take x_4 and x_5 as parameters, and so we have that the infinitely many solutions to the system are given by

$$x_1 = x_2 = \frac{s}{3} + \frac{2t}{3} - 1, \quad x_3 = -\frac{5s}{3} - \frac{7t}{3} + 3, \quad x_4 = s, \quad x_5 = t,$$

for parameters s, t.

Now, to get a different expression for the solutions, let us reduce the rows in a different way. To work a little less, note that from the original system $x_2 = x_1$. So we can look at the system

$$\begin{cases} 3x_1 + 3x_3 + 4x_4 + 5x_5 = 6 \\ 4x_1 + 5x_3 + 7x_4 + 9x_5 = 11 \end{cases}$$

and use $x_2 = x_1$ at the end. Also as we want a different parametrization, let us clear the third and four columns first. Note a way to avoid fractions.

$$\left[\begin{array}{cccc|c} 3 & 3 & 4 & 5 & 6 \\ 4 & 5 & 7 & 9 & 11 \end{array}\right]\xrightarrow[5R2]{9R1}\left[\begin{array}{cccc|c} 27 & 27 & 36 & 45 & 54 \\ 20 & 25 & 35 & 45 & 55 \end{array}\right]\xrightarrow{R1-R2}\left[\begin{array}{cccc|c} 7 & 2 & 1 & 0 & -1 \\ 20 & 25 & 35 & 45 & 55 \end{array}\right]$$

$$\xrightarrow{\frac{1}{5}R2}\left[\begin{array}{cccc|c} 7 & 2 & 1 & 0 & -1 \\ 4 & 5 & 7 & 9 & 11 \end{array}\right]\xrightarrow{R2-7R1}\left[\begin{array}{cccc|c} 7 & 2 & 1 & 0 & -1 \\ -45 & -9 & 0 & 9 & 18 \end{array}\right]$$

$$\xrightarrow{\frac{1}{9}R2}\left[\begin{array}{cccc|c} 7 & 2 & 1 & 0 & -1 \\ -5 & -1 & 0 & 1 & 2 \end{array}\right]$$

This reads $x_4 = -7x_1 + 2x_3 - 1$, $x_5 = 5x_1 + x_3 + 2$. Taking $x_1 = s$, $x_3 = t$ as parameters, we get the solutions

$$x_1 = s, \quad x_2 = s, \quad x_3 = t, \quad x_4 = -7s - 2t - 1, \quad x_5 = 5s + t + 2.$$

2. In each case, write a system of linear equations consisting of three equations in three unknowns, with all coefficients of all the variables different from zero, and with

(a) no solution;

Answer. Let us start with an echelon form with no solution and do some (mostly random) row operations to get less obvious system.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1+3R2+8R3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2+2R1+4R3} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 8 \\ 2 & 7 & 2 & 16 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3+R1+R2} \left[\begin{array}{ccc|c} 1 & 3 & 1 & 8 \\ 2 & 7 & 2 & 16 \\ 3 & 10 & 3 & 25 \end{array}\right]$$

Thus the system

$$\begin{cases} x + 3y + z = 8 \\ 2x + 7y + 2z = 16 \\ 3x + 10y + 3z = 25 \end{cases}$$

is an example of a linear system of three equations on three unknowns with no solution.

(b) exactly one solution;

Answer. We use the same idea, but now we start with say $x = 1$, $y = 2$, $z = 3$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R1+3R2+8R3} \left[\begin{array}{ccc|c} 1 & 3 & 8 & 31 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{R2+R1} \left[\begin{array}{ccc|c} 1 & 3 & 8 & 31 \\ 1 & 4 & 8 & 33 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R3+2R1} \left[\begin{array}{ccc|c} 1 & 3 & 8 & 31 \\ 1 & 4 & 8 & 33 \\ 2 & 6 & 17 & 65 \end{array}\right]$$

Thus the system

$$\begin{cases} x + 3y + 8z = 31 \\ x + 4y + 8z = 33 \\ 2x + 6y + 17z = 65 \end{cases}$$

has unique solution $x = 1$, $y = 2$, $z = 3$.

(c) infinitely many solutions.

Answer. We start with a row of zeros. We also include z in the first two equations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R1+3R2} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R2+2R1} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 2 & 7 & 9 & 16 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R3+R2+R1} \left[\begin{array}{ccc|c} 1 & 3 & 4 & 7 \\ 2 & 7 & 9 & 16 \\ 3 & 10 & 13 & 23 \end{array}\right]$$

So the system

$$\begin{cases} x + 3y + 4z = 7 \\ 2x + 7y + 9z = 16 \\ 3x + 10y + 13z = 23 \end{cases}$$

has infinitely many solutions.

3. Consider the system

$$\begin{cases} 2x - 3y = 1 \\ x + y = 2 \\ x + 3y = a \end{cases}.$$

Find the value of a that makes the system consistent.

Answer. We perform row reduction:

$$\left[\begin{array}{cc|c} 2 & -3 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & a \end{array}\right] \xrightarrow[R2-R3]{R1-2R3} \left[\begin{array}{cc|c} 0 & -9 & 1-2a \\ 0 & -2 & 2-a \\ 1 & 3 & a \end{array}\right] \xrightarrow[-\frac{1}{2}R2]{-\frac{1}{9}R1} \left[\begin{array}{cc|c} 0 & 1 & (2a-1)/9 \\ 0 & 1 & (a-2)/2 \\ 1 & 3 & a \end{array}\right]$$

$$\xrightarrow[R3-3R2]{R1-R2} \left[\begin{array}{cc|c} 0 & 0 & (2a-1)/9-(a-2)/2 \\ 0 & 1 & (a-2)/2 \\ 1 & 0 & a-3(a-2)/2 \end{array}\right] \xrightarrow{R1\leftrightarrow R3} \left[\begin{array}{cc|c} 1 & 0 & a-3(a-2)/2 \\ 0 & 1 & (a-2)/2 \\ 0 & 0 & (2a-1)/9-(a-2)/2 \end{array}\right]$$

Now note that the last equation is $0 = (2a-1)/9 - (a-2)/2$. For this to hold, we need $(2a-1)/9 = (a-2)/2$, that is $a = 16/5$. With that choice the system becomes a valid reduced row echelon form which has a unique solution.

Suggested practice questions: 1.1: 1-14.