Math 122-002 201730
Practice Assignment # 3 – Answers

1. The scores of three players in a tournament have been lost. The only information available is that the sum of score for players 1 and 2 is 20, the sum of the scores of players 2 and 3 is 15, and the sum of the scores for players 1 and 3 is 17.

   (a) Show that the individual scores can be recovered. Find them!

   (b) If we had four players, and we know the combined scores of 1 and 2, 2 and 3, 3 and 4, and 4 and 1, can the scores be recovered?

Answer.

(a) If the scores are x, y, z respectively, we have

$$\begin{cases} x + y \quad\quad = 20 \\ \quad\; y + z = 15 \\ x \quad\; + z = 17 \end{cases}$$

Working on the augmented matrix,

$$\begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & 15 \\ 1 & 0 & 1 & 17 \end{bmatrix} \xrightarrow{R3-R1} \begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & 15 \\ 0 & -1 & 1 & -3 \end{bmatrix} \xrightarrow{R3+R2} \begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 2 & 12 \end{bmatrix} \xrightarrow{\frac{1}{2}R3} \begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

$$\xrightarrow{R2-R3} \begin{bmatrix} 1 & 1 & 0 & 20 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 6 \end{bmatrix} \xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

So the scores were 11, 9, and 6 respectively.

(b) If we denote the scores of the four players by a, b, c, d, we get the system

$$\begin{cases} x_1 + x_2 = a \\ x_2 + x_3 = b \\ x_3 + x_4 = c \\ x_1 + x_4 = d \end{cases}$$

Working on the augmented matrix,

$$\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 1 & 0 & 0 & 1 & d \end{bmatrix} \xrightarrow{R4-R1} \begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & -1 & 0 & 1 & d-a \end{bmatrix} \xrightarrow{R4+R2} \begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 1 & 1 & d-a+b \end{bmatrix}$$

$$\xrightarrow{R4-R3} \begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 0 & 0 & 1 & 1 & c \\ 0 & 0 & 0 & 0 & d-a+b-c \end{bmatrix}$$

For this system to be consistent, we need $d-a+b-c=0$, or $a-d=b-c$. If the scores satisfy that relation, then the system is consistent and can be solved.

2. Find a such that the system has nontrivial solutions. $\left\{ \begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0 \\ x & + & ay & - & 3z & = & 0 \\ -x & + & 6y & - & 5z & = & 0 \end{array} \right.$

Answer. Working on the augmented matrix of the system (as usual with homogeneous systems, we can omit the independent column since it only consists of zeroes; one needs to be careful though, to remember that it is still there). For the system to have nontrivial solution, we need the row echelon form to have less than three leading ones (because if there are three, then the only solution is the trivial one $x=y=z=0$).

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & a & -3 \\ -1 & 6 & -5 \end{bmatrix} \xrightarrow{R3+R1} \begin{bmatrix} 1 & 2 & 1 \\ 1 & a & -3 \\ 0 & 8 & -4 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & a-2 & -4 \\ 0 & 8 & -4 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & a-2 & -4 \\ 0 & 10-a & 0 \end{bmatrix}.$$

We see that if $a=10$, we get a row of zeroes, and so the system will have infinitely many solution (homogeneous, three variables, two equations).

3. Without solving the system ("by inspection"), determine whether each homogeneous system has nontrivial (that is, nonzero) solutions. Justify your answers.

(a) $\left\{ \begin{array}{rcrcrcrcl} 2x_1 & + & 2x_2 & + & 2x_3 & + & 5x_4 & = & 0 \\ -x_1 & + & 5x_2 & + & 2x_3 & - & 6x_4 & = & 0 \\ 8x_1 & + & x_2 & + & 4x_3 & + & 1000x_4 & = & 0 \end{array} \right.$

(b) $\begin{cases} x_1 + 5x_2 - 2x_3 = 0 \\ 5x_2 + 8x_3 = 0 \\ 2x_3 = 0 \end{cases}$

(c) $\begin{cases} 2x_1 + 2x_2 = 0 \\ -x_1 - x_2 = 0 \end{cases}$

Answer.

(a) This system is homogeneous, three equations, four variables. So it has infinitely many solutions.

(b) Dividing the second row by 5 and the third one by 2 gives us three leading ones. So the system will have unique solution: the third equation implies that $x_3 = 0$; then we look at the second one, and $x_2 = 0$, and now the first one gives $x_1 = 0$.

(c) The second equation is a multiple of the first one, so this is in essence a single linear equation in two variables, and thus has infinitely many solutions.

4. Suppose that A, B, C, D are matrices of sizes 3×4, 2×3, 3×3 and 2×4 respectively. Decide whether the given expression is defined; if it is, give the size of the resulting matrix

(a) AB

(b) BA

(c) $BA + C$

(d) BD

(e) DAC

(f) $D + B$

Answer.

(a) Since A has 4 columns and B has 2 rows, AB is not defined.

(b) BA is going to be 2×4

(c) Since BA is 2×4 and C is 3×3, the sum is not defined.

(d) B has 3 columns and D has 2 rows, so BD is not defined.

(e) D has 4 columns and A has 3 rows, so DA is not defined.

(f) D and B are different sizes, so the sum is not defined.

5. The *transpose* of an $m \times n$ matrix A is the matrix A^T obtained by interchanging the rows and columns of A. This means that the first row of A^T is the first column of A, etc. In symbols, $(A^T)_{ij} = A_{ji}$.

   Now consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 7 \\ 5 & 2 & 0 \\ 1 & 2 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 6 & 3 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

   and compute the given expression where possible:

   (a) $AB + C^T$
   (b) $(BA)^T$
   (c) ACB
   (d) $(B + C)A$
   (e) $A(B + C)$

Answer.

(a) A has 2 columns and B has 3 rows, so AB is not defined.

(b)

$$BA = \begin{bmatrix} 4 \times 2 + 1 \times 1 + 7 \times 3 & 4 \times 0 + 1 \times 3 + 7 \times (-1) \\ 5 \times 2 + 2 \times 1 + 0 \times 3 & 5 \times 0 + 2 \times 3 + 0 \times (-1) \\ 1 \times 2 + 2 \times 1 + 5 \times 3 & 1 \times 0 + 2 \times 3 + 5 \times (-1) \end{bmatrix} = \begin{bmatrix} 30 & -4 \\ 12 & 6 \\ 19 & 1 \end{bmatrix}.$$

Then

$$(BA)^T = \begin{bmatrix} 30 & 12 & 19 \\ -4 & 6 & 1 \end{bmatrix}.$$

(c) AC is not possible, since A has 2 columns and C has 3 rows.

(d)

$$B + C = \begin{bmatrix} 4 & 1 & 7 \\ 5 & 2 & 0 \\ 1 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 6 & 3 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 10 \\ 7 & 3 & 3 \\ 2 & 2 & 6 \end{bmatrix}$$

$$(B + C)A = \begin{bmatrix} 8 & 7 & 10 \\ 7 & 3 & 3 \\ 2 & 2 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & 0 \\ 1 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 53 & 11 \\ 26 & 6 \\ 24 & 0 \end{bmatrix}$$

(e) A is 3×2 and $B + C$ is 3×3, so the product is not defined: A has 2 columns and $B + C$ has 3 rows.