Math 122-002 201730
Practice Assignment # 4 – Answers

1. Let A be an $m \times n$ matrix such that A^2 can be formed. What can you say about m and n?

   Answer. For $A^2 = A \times A$ to be formed, we need n (the number of columns of the first A) to be equal to m (the number of rows of the second A). So A is a square matrix.

2. Find A in terms of B if $A - B = 5A + 3B$.

   Answer. We have $A - B = 5A + 3B$. If we subtract A from both sides:
   $$-B = A - B - a = 5A + 3B - A = 4A + 3B.$$
   If we subtract $3B$ from both sides:
   $$-4B = 4A + 3B - 3B = 4A.$$
   If we know multiply both sides by $1/4$, we get $A = -B$.

3. Suppose that X, Y, A, B are matrices of the same size. If $4X + 3Y = A$ and $2X + 2Y = B$, write X and Y in terms of A and B.

   Answer. If we multiply the second equation by 2, we have
   $$4X + 3Y = A$$
   $$4X + 4Y = 2B$$
   Subtracting the first equation from the second one, $Y = 2B - A$. Now the first equation becomes
   $$4X + 3(2B - A) = A.$$
   Solving for X,
   $$X = \frac{1}{4}(4A - 6B) = A - \frac{3}{2}B, \qquad Y = B - A.$$

4. Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some numbers a, b, c, d. Show that

(a) if A commutes with $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $b = c = 0$;

(b) if A commutes with $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $c = 0$, $a = d$.

Answer.

(a) We want

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Calculating the products we get the equality

$$\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$$

Looking at the $2,1$ and $1,2$ entries we see that $b = c = 0$.

(b) We want

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Calculating the products we get the equality

$$\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

Looking at the $1,2$ entries we see that $a = d$. And from the $1,1$ (or $2,2$) entry we get $c = 0$.

5. Let

$$\vec{a_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{a_2} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{a_3} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad \vec{a_4} = \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}.$$

In each case express $\vec{b}$ as a linear combination of $\vec{a_1}$, $\vec{a_2}$, $\vec{a_3}$, $\vec{a_4}$, or show that such a linear combination doesn't exist.

(a) $\vec{b} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$

(b) $\vec{b} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$

Answer.

(a) The existence of the linear combination depends on the consistency of the system $A\vec{x} = \vec{b}$. Doing row reduction, with

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 1 & 0 & -1 & -3 \\ -1 & 2 & 3 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 0 \\ 1 & 0 & -1 & -3 & 3 \\ -1 & 2 & 3 & 5 & 5 \end{array}\right] \xrightarrow[R3+R1]{R2-R1} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 0 \\ 0 & -3 & -3 & -3 & 3 \\ 0 & 5 & 5 & 5 & 5 \end{array}\right] \xrightarrow[\frac{1}{5}R3]{-\frac{1}{3}R2} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$
$$\xrightarrow{R3-R2} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

We see from the last row that the system is inconsistent, and this implies that $\vec{b}$ cannot be written as a linear combination of $\vec{a_1}$, $\vec{a_2}$, $\vec{a_3}$, $\vec{a_4}$.

(b) Again the existence of the linear combination depends on the consistency of the system $A\vec{x} = \vec{b}$. Doing row reduction, with

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 1 & 0 & -1 & -3 \\ -1 & 2 & 3 & 5 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 4 \\ 1 & 0 & -1 & -3 & 1 \\ -1 & 2 & 3 & 5 & 1 \end{array}\right] \xrightarrow[R3+R1]{R2-R1} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 4 \\ 0 & -3 & -3 & -3 & -3 \\ 0 & 5 & 5 & 5 & 5 \end{array}\right] \xrightarrow[\frac{1}{5}R3]{-\frac{1}{3}R2}$$
$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$
$$\xrightarrow{R3-R2} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R1-3R2} \left[\begin{array}{cccc|c} 1 & 0 & -1 & -3 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

As a solution to the system, this tells us that we can take $x_3 = s$, $x_4 = t$, and then $x_1 = 1 + s + 3t$, $x_2 = 1 - s - t$. Any choice of s, t will do, so we can as well take $s = t = 0$. Then $x_1 = x_2 = 1$, $x_3 = x_4 = 0$. That is, $\vec{b} = \vec{a_1} + \vec{a_2}$.

There are infinitely many solutions, so we could also take other values of s, t. If for instance $s = 1$, $t = 2$, then $x_1 = 8$, $x_2 = -2$, $x_3 = 1$, $x_4 = 2$. So we have

$$\vec{b} = 8\vec{a_1} - 2\vec{a_2} + \vec{a_3} + 2\vec{a_4}.$$