Math 122-002 201730

Practice Assignment # 6 – Answers

1. Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix. Do the product to check that the elementary matrix actually performs the operation.

(a) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$

Answer.

(a) This matrix was obtained from the identity by $R1 \leftrightarrow R3$. The reverse operation is also $R1 \leftrightarrow R3$. So

$$E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$EA = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) This matrix was obtained from the identity by $R1 + 8R3$. The reverse operation is $R1 - 8R3$, and its elementary matrix is

$$E = \begin{bmatrix} 1 & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$EA = \begin{bmatrix} 1 & 0 & -8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) This matrix was obtained from the identity by $R2 \leftrightarrow R3$. The reverse operation is also $R2 \leftrightarrow R3$. So

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) This matrix was obtained from the identity by $R3 - 1/2\,R1$. The reverse operation is $R3 + 1/2\,R1$, and its elementary matrix is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}.$$

Then

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Use the inversion algorithm to find the inverse where possible. If the matrix is invertible, write the inverse as a product of elementary matrices.

(a) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 0 & 3 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 \\ 2 & 1 & 5 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 2 & 4 & 6 & 0 \\ 2 & 4 & 6 & 8 \end{bmatrix}$

Answer.

(a) A matrix with a row of zeros cannot be invertible (its product with any other matrix will have a row of zeros, and so it cannot be the identity).

(b)

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 5 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R2\leftrightarrow R3}\begin{bmatrix}1 & 0 & 0 & 1 & | & 0 & 1 & 0 & 0\\0 & -2 & 3 & 0 & | & 0 & 0 & 1 & 0\\0 & 0 & 2 & 0 & | & 1 & 0 & 0 & 0\\2 & 1 & 5 & 3 & | & 0 & 0 & 0 & 1\end{bmatrix}\xrightarrow{R4-2R1}\begin{bmatrix}1 & 0 & 0 & 1 & | & 0 & 1 & 0 & 0\\0 & -2 & 3 & 0 & | & 0 & 0 & 1 & 0\\0 & 0 & 2 & 0 & | & 1 & 0 & 0 & 0\\0 & 1 & 5 & 1 & | & 0 & -2 & 0 & 1\end{bmatrix}$$

$$\xrightarrow[-1/2R2]{1/2R3}\begin{bmatrix}1 & 0 & 0 & 1 & | & 0 & 1 & 0 & 0\\0 & 1 & -3/2 & 0 & | & 0 & 0 & -1/2 & 0\\0 & 0 & 1 & 0 & | & 1/2 & 0 & 0 & 0\\0 & 1 & 5 & 1 & | & 0 & -2 & 0 & 1\end{bmatrix}$$

$$\xrightarrow{R4-R2}\begin{bmatrix}1 & 0 & 0 & 1 & | & 0 & 1 & 0 & 0\\0 & 1 & -3/2 & 0 & | & 0 & 0 & -1/2 & 0\\0 & 0 & 1 & 0 & | & 1/2 & 0 & 0 & 0\\0 & 0 & 13/2 & 1 & | & 0 & -2 & 1/2 & 1\end{bmatrix}$$

$$\xrightarrow{R4-13/2R3}\begin{bmatrix}1 & 0 & 0 & 1 & | & 0 & 1 & 0 & 0\\0 & 1 & -3/2 & 0 & | & 0 & 0 & -1/2 & 0\\0 & 0 & 1 & 0 & | & 1/2 & 0 & 0 & 0\\0 & 0 & 0 & 1 & | & -13/4 & -2 & 1/2 & 1\end{bmatrix}$$

$$\xrightarrow{R2+3/2R3}\begin{bmatrix}1 & 0 & 0 & 1 & | & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & | & 3/4 & 0 & -1/2 & 0\\0 & 0 & 1 & 0 & | & 1/2 & 0 & 0 & 0\\0 & 0 & 0 & 1 & | & -13/4 & -2 & 1/2 & 1\end{bmatrix}$$

$$\xrightarrow{R1-R4}\begin{bmatrix}1 & 0 & 0 & 0 & | & 13/4 & 3 & -1/2 & -1\\0 & 1 & 0 & 0 & | & 3/4 & 0 & -1/2 & 0\\0 & 0 & 1 & 0 & | & 1/2 & 0 & 0 & 0\\0 & 0 & 0 & 1 & | & -13/4 & -2 & 1/2 & 1\end{bmatrix}$$

Collecting the row operations we did, we can write A^{-1} as a product of elementary matrices:

$$A^{-1}=\begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 3/2 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & -13/2 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & -1 & 0 & 1\end{bmatrix}$$
$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & -1/2 & 0 & 0\\0 & 0 & 1/2 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\-1 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

(c) We form the augmented matrix and we perform the row reduction, keeping track of the operations:

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 2 & 1 & 3 & | & 0 & 1 & 0 \\ 0 & 3 & 3 & | & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -3 & -3 & | & -2 & 1 & 0 \\ 0 & 3 & 3 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R3+R2} \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & -3 & -3 & | & -2 & 1 & 0 \\ 0 & 0 & 0 & | & -2 & 1 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{3}R2} \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 1 & | & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & | & -2 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R1-2R2} \begin{bmatrix} 1 & 0 & 1 & | & -1/3 & 2/3 & 0 \\ 0 & 1 & 1 & | & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & | & -2 & 1 & 1 \end{bmatrix}$$

So A is not invertible.

(d) Same drill:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 2 & 4 & 6 & 0 & | & 0 & 0 & 1 & 0 \\ 2 & 4 & 6 & 8 & | & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R4-R3} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & | & 0 & 1 & 0 & 0 \\ 2 & 4 & 6 & 0 & | & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 & | & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R3-R2} \begin{bmatrix} 1 & 0 & 0 & 0| & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0| & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0| & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 8| & 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & | & -2 & 1 & 0 & 0 \\ 0 & 0 & 6 & 0 & | & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 8 & | & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{1/8R4 \\ 1/6R3 \\ 1/4R2}} \begin{bmatrix} 1 & 0 & 0 & 0| & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0| & -1/2 & 1/4 & 0 & 0 \\ 0 & 0 & 1 & 0| & 0 & -1/6 & 1/6 & 0 \\ 0 & 0 & 0 & 1| & 0 & 0 & -1/8 & 1/8 \end{bmatrix}$$

Collecting the row operations in terms of the elementary matrices:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1/4 & 0 & 0 \\ 0 & -1/6 & 1/6 & 0 \\ 0 & 0 & -1/8 & 1/8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/6 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

3. Find the value of c for which A has no inverse.

(a) $A = \begin{bmatrix} 1 & -2 \\ 3 & c \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & -1 & c \\ 0 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$.

Answer.

(a) The matrix A will fail to be invertible precisely when $1 \times c - 3 \times (-2) = 0$, i.e. when $c + 6 = 0$. So $c = -6$. We can also see this through row-reduction: if we do $R2 - 3R1$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & c+6 \end{bmatrix}.$$

The only way a leading one can appear in the second row is when $c + 6 \neq 0$.

(b) Again we look at row reduction. If we perform $R3 + R1$ and then $R3 + \frac{1}{2}R2$, we get

$$\begin{bmatrix} 1 & -1 & c \\ 0 & 2 & 1 \\ 0 & -1 & 1+c \end{bmatrix} \to \begin{bmatrix} 1 & -1 & c \\ 0 & 2 & 1 \\ 0 & 0 & \frac{3}{2}+c \end{bmatrix}.$$

For the third row to have a leading one, we need $\frac{3}{2} + c \neq 0$. So A will have no inverse when $c = -3/2$.

4. Show that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Answer.

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & db-bd \\ -ac+ac & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

5. Use the inverse matrix to solve the system

(a) $\begin{cases} 3x - 2y = 5 \\ x + 4y = 6 \end{cases}$

(b) $\begin{cases} x + 4y + 2z = 1 \\ 2x + 3y + 3z = -1 \\ 4x + y + 4z = 0 \end{cases}$

Answer.

(a) From the previous question we know that

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}^{-1} = \frac{1}{14} \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}.$$

So

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 32 \\ 13 \end{bmatrix},$$

i.e. $x = 16/7$, $y = 13/14$.

(b) First we find the inverse:

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 2 & 3 & 3 & 0 & 1 & 0 \\ 4 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R3-4R1]{R2-2R1} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & -15 & -4 & -4 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3-3R2} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & 2 & -3 & 1 \end{array}\right] \xrightarrow[-R3]{R2-R3} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & -5 & 0 & -4 & 4 & -1 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R2} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4/5 & -4/5 & 1/5 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{array}\right] \xrightarrow{R1-4R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -11/5 & 16/5 & -4/5 \\ 0 & 1 & 0 & 4/5 & -4/5 & 1/5 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{array}\right]$$

$$\xrightarrow{R1-2R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/5 & -14/5 & 6/5 \\ 0 & 1 & 0 & 4/5 & -4/5 & 1/5 \\ 0 & 0 & 1 & -2 & 3 & -1 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} 9/5 & -14/5 & 6/5 \\ 4/5 & -4/5 & 1/5 \\ -2 & 3 & -1 \end{bmatrix}.$$

Solving the system: $\vec{x} = A^{-1}\vec{b}$, that is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9/5 & -14/5 & 6/5 \\ 4/5 & -4/5 & 1/5 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 23/5 \\ 8/5 \\ -5 \end{bmatrix}$$

6. Find A if $(I + 3A)^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$.

Answer. If $(I + 3A)^{-1} = B$, then $I + 3A = B^{-1}$, or $A = (B^{-1} - I)/3$. Here,

$$B^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & -1 \end{bmatrix}.$$

Then

$$A = (B^{-1} - I)/3 = \frac{1}{3} \begin{bmatrix} 1/2 - 1 & 0 \\ 1/2 & -1 - 1 \end{bmatrix} = \begin{bmatrix} -1/6 & 0 \\ 1/6 & -2/3 \end{bmatrix}$$