Math 122-002 201730

Practice Assignment # 7 – Answers

Please remember that the assignment consists of only a sample of the kind of questions you are supposed to be able to do. It is **not** *a safe practice to just do the assignment, and that is why there is a list of "suggested practice problems".*

1. Calculate the determinant by the first column.

(a) $\begin{bmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 2 & -1 \\ -3 & 5 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{bmatrix}$

Answer. Assume that in each question we call the matrix A.

(a)

$$\det A = 2\begin{vmatrix} 5 & 2 \\ 1 & 4 \end{vmatrix} - (-2)\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} + 8\begin{vmatrix} 2 & 2 \\ 5 & 2 \end{vmatrix} = 2 \times 18 + 2 \times 6 + 8 \times (-6) = 0.$$

(b)

$$\det A = 1\begin{vmatrix} 2 & -1 \\ 5 & -4 \end{vmatrix} - 3\begin{vmatrix} 3 & -2 \\ 5 & -4 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -2 \\ 2 & -1 \end{vmatrix} = 1 \times (-3) - 3 \times (-2) - 3 \times (1) = 0.$$

(c)

$$\begin{aligned}
\det A &= 0 \times \begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 & 4 \\ 0 & 1 & 2 \\ 2 & 1 & 1 \end{vmatrix} \\
&= 0 - \left(2\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}\right) + 1\left(2\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}\right) \\
&- 2\left(2\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix}\right) \\
&= -(2 \times 0 - 2 \times (-4) + (-2)) + (2 \times (-2) - 0 + 0) - 2(2 \times (-1) - 0 + 2 \times 0) \\
&= -6 - 4 + 4 = -6.
\end{aligned}$$

2. For the same matrices as in the previous question, calculate the determinant by row/column reduction.

Answer.

(a) We do $R2 + R1$, $R3 - 4R1$. In the second step, $R3 + R2$ (none of this operations affect the determinant).

$$\begin{vmatrix} 2 & 2 & 2 \\ -2 & 5 & 2 \\ 8 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 \\ 0 & 7 & 4 \\ 0 & -7 & -4 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 \\ 0 & 7 & 4 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Now the determinant has a zero row, so we know it is zero.

(b) First we do $R3 + R2$ and $R2 - 3R1$; then $R3 + R2$.

$$\begin{vmatrix} 1 & 3 & -2 \\ 3 & 2 & -1 \\ -3 & 5 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -7 & 5 \\ 0 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 \\ 0 & -7 & 5 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

since the last row is zero.

(c) First we do $C3 - C1$ and $C4 - C1$; then $C2 - C3$ and $C4 - 2C3$; finally, $C1 - C4$ and (for clarity) $C1 \leftrightarrow C3$, $C2 \leftrightarrow C4$.

$$\begin{vmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{vmatrix}$$

$$= 2 \times 1 \times \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 2 \times (-3) = -6.$$

3. Find the inverse using the adjoint formula

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & 2 \\ 1 & 0 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

Answer.

(a) Since A is triangular, $\det A = 1 \times 3 \times 6 = 18$.

$$A^{-1} = \frac{1}{18} \begin{bmatrix} \begin{vmatrix} 3 & 0 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 4 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} \\ -\begin{vmatrix} 0 & 0 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 4 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 4 & 5 \end{vmatrix} \\ \begin{vmatrix} 0 & 0 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \end{bmatrix}^T = \frac{1}{18} \begin{bmatrix} 18 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ -1/9 & -5/18 & 1/6 \end{bmatrix}$$

(b) First, $\det B = -(2-6) + 2 \times 2 = 8$. Now

$$B^{-1} = \frac{1}{8}\begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} \\ \\ -\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix} \\ \\ \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T = \frac{1}{8}\begin{bmatrix} -3 & 1 & 3 \\ 4 & -4 & 4 \\ 2 & 2 & -2 \end{bmatrix}^T = \begin{bmatrix} -3/8 & 1/2 & 1/4 \\ 1/8 & -1/2 & 1/4 \\ 3/8 & 1/2 & -1/4 \end{bmatrix}$$

4. For each matrix in the previous question, use Cramer's rule to solve the system

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Answer.

(a)

$$x = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 5 & 6 \end{vmatrix}}{\det A} = \frac{18}{18} = 1.$$

$$y = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 4 & 3 & 6 \end{vmatrix}}{\det A} = \frac{0}{18} = 0.$$

$$z = \frac{\begin{vmatrix} 1 & 0 & 1 \\ 2 & 3 & 2 \\ 4 & 5 & 3 \end{vmatrix}}{\det A} = \frac{-1 + (-2)}{18} = \frac{(-3)}{18} = -\frac{1}{6}.$$

(b)

$$x = \frac{\begin{vmatrix} 1 & 2 & 2 \\ 2 & 0 & 1 \\ 3 & 3 & 1 \end{vmatrix}}{\det A} = \frac{-2(-4) - (-3)}{8} = \frac{11}{8}.$$

$$y = \frac{\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix}}{\det A} = \frac{-1(-1) + 2(-1)}{8} = -\frac{1}{8}.$$

$$z = \frac{\begin{vmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 3 & 3 \end{vmatrix}}{\det A} = \frac{-2(-1)+3}{8} = \frac{5}{8}.$$

5. Evaluate the determinant "by inspection" (that is, you should be able to do it without writing any computations).

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 4 & 5 & 8 \end{bmatrix}$

Answer.

(a) This is a triangular matrix, so the determinant is the product of the diagonal entries, that is $1 \times 3 \times 6 = 18$.

(b) An quick way here is to notice that the third column is two times the first one. This tells us that a row operation (C3-2C1) will make the last column equal to zero. So the determinant is zero.

We could also say that because the last column is twice the first column, we can factor the 2 from the third column and the remaining matrix has two equal columns, which makes the determinant equal to zero.

6. Evaluate the determinant by first multiplying and dividing by a suitable number to eliminate the fractions.

(a) $\begin{bmatrix} 1/2 & 1/3 & 1/4 \\ 2/5 & 3/7 & 0 \\ 4/7 & 5/8 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 1/3 & 0 & 2/9 \\ 2/7 & 3/11 & 4 \\ 4 & -5/3 & 8 \end{bmatrix}$

Answer.

(a)

$$\begin{vmatrix} 1/2 & 1/3 & 1/4 \\ 2/5 & 3/7 & 0 \\ 4/7 & 5/8 & 6 \end{vmatrix} = \frac{1}{12} \times 12 \begin{vmatrix} 1/2 & 1/3 & 1/4 \\ 2/5 & 3/7 & 0 \\ 4/7 & 5/8 & 6 \end{vmatrix} = \frac{1}{12} \begin{vmatrix} 6 & 4 & 3 \\ 2/5 & 3/7 & 0 \\ 4/7 & 5/8 & 6 \end{vmatrix} = \frac{1}{12} \times \frac{1}{35} \begin{vmatrix} 6 & 4 & 3 \\ 14 & 15 & 0 \\ 4/7 & 5/8 & 6 \end{vmatrix}$$

$$= \frac{1}{12} \times \frac{1}{35} \times \frac{1}{56} \begin{vmatrix} 6 & 4 & 3 \\ 14 & 15 & 0 \\ 32 & 35 & 336 \end{vmatrix}$$

$$= \frac{1}{12 \times 35 \times 56} [3 \times (14 \times 35 - 32 \times 15) + 336(6 \times 15 - 14 \times 4)]$$

$$= \frac{30 + 336 \times 34}{12 \times 35 \times 56} = \frac{11454}{12 \times 35 \times 56} = \frac{11454}{23520} = \frac{1909}{3920}.$$

(b) (the column operations we will do are C1-C3 and C3-2C1)

$$\begin{vmatrix} 1/3 & 0 & 2/9 \\ 2/7 & 3/11 & 4 \\ 4 & -5/3 & 8 \end{vmatrix} = \frac{1}{9} \times \frac{1}{77} \times \frac{1}{3} \begin{vmatrix} 3 & 0 & 2 \\ 22 & 21 & 308 \\ 12 & -5 & 24 \end{vmatrix} = \frac{1}{2079} \begin{vmatrix} 1 & 0 & 2 \\ -286 & 21 & 308 \\ -12 & -5 & 24 \end{vmatrix}$$

$$= \frac{1}{2079} \begin{vmatrix} 1 & 0 & 0 \\ -286 & 21 & 880 \\ -12 & -5 & 48 \end{vmatrix} = \frac{21 \times 48 + 5 \times 880}{2079} = \frac{5408}{2079}$$