Math 122-002 201730

Practice Assignment # 8 – Answers

Please remember that the assignment consists of only a sample of the kind of questions you are supposed to be able to do. It is **not** *a safe practice to just do the assignment, and that is why there is a list of "suggested practice problems".*

1. In each case find the characteristic polynomial, eigenvalues, eigenvectors, and—if possible—an invertible matrix P such that $P^{-1}AP$ is diagonal.

   (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

   Answer. We begin with the characteristic polynomial:

   $$p(\lambda) = \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)-6 = \lambda^2-3\lambda-4 = (\lambda-4)(\lambda+1)$$

   (if you don't see this last factorization, you just use the quadratic formula to find that the roots are 4 and -1). So the eigenvalues are 4 and -1.

   Next we look for the eigenvectors for the eigenvalue 4. These are those satisfying $(A-4I)\vec{x} = \vec{0}$. By row reduction:

   $$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \xrightarrow{R2+R1} \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix}$$

   So the eigenvectors for 4 are those vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ with $-3x+2y = 0$. They will be multiples of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ (taking $y = 3$).

   Eigenvectors for -1:

   $$\begin{bmatrix} [c]1-(-1) & 2 \\ 3 & 2-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \xrightarrow{R2-\frac{3}{2}R1} \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

   The eigenvectors for -1 are then those satisfying $x+y = 0$. These are the multiplies of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   As all eigenvalues are nonzero, we know that A is invertible. Then we form P with one eigenvector for each eigenvalue in its columns:

   $$P = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}.$$

(b) $A = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}$.

Answer. Characteristic polynomial: calculating by the second column,

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & 0 & -4 \\ 0 & 5-\lambda & 0 \\ 5 & 0 & -2-\lambda \end{vmatrix} = (5-\lambda)(20-(7-\lambda)(2+\lambda))$$

$$= (5-\lambda)(\lambda^2 - 5\lambda + 6) = (5-\lambda)(\lambda-2)(\lambda-3).$$

So the eigenvalues are $2, 3, 5$. We look for eigenvectors now.

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 3 & 0 \\ 5 & 0 & -4 \end{bmatrix} \xrightarrow{R3-R1} \begin{bmatrix} 5 & 0 & -4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $x_2 = 0$ and, with $x_3 = t$, $5x_1 = 4t$. The eigenvectors for 2 are of then of the form

$$\begin{bmatrix} 4t/5 \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 4/5 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ 5 & 0 & -5 \end{bmatrix} \xrightarrow{R3-5/4R1} \begin{bmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $x_2 = 0$, and $x_1 = x_3$. Thus the eigenvectors are of the form

$$t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda = 5$,

$$A - 5I = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & 0 \\ 5 & 0 & -7 \end{bmatrix} \xrightarrow{R3-5/2R1} \begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow[R1+4R3]{1/3R3} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so $x_1 = x_3 = 0$. The eigenvectors for 5 are then of the form

$$t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

One matrix P that diagonalizes A is then

$$P = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix}.$$

So

$$P^{-1} = \begin{bmatrix} -1 & 5 & 0 \\ 0 & 0 & 1 \\ 1 & -4 & 0 \end{bmatrix}^T = \begin{bmatrix} -1 & 0 & 1 \\ 5 & 0 & -4 \\ 0 & 1 & 0 \end{bmatrix}.$$

and

$$\begin{bmatrix} 4 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \\ 5 & 0 & -4 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 & -4 \\ 0 & 5 & 0 \\ 5 & 0 & -2 \end{bmatrix}.$$

2. Diagonalize A and use this information to find A^{10}.

(a)

$$A = \begin{bmatrix} -6 & 11 & -4 \\ -8 & 13 & -4 \\ -11 & 17 & -5 \end{bmatrix}.$$

Answer. Characteristic polynomial: by brute-force calculation,

$$\det(A - \lambda I) = \begin{vmatrix} -6-\lambda & 11 & -4 \\ -8 & 13-\lambda & -4 \\ -11 & 17 & -5-\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2.$$

By inspection we see that $\lambda = -1$ is a root. Long division then shows that

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = (\lambda + 1)(-\lambda^2 + 3\lambda - 2) = -(\lambda + 1)(\lambda - 2)(\lambda - 1).$$

The eigenvalues are -1, 2, and 1.
For $\lambda = -1$,

$$A + \lambda I = \begin{bmatrix} -5 & 11 & -4 \\ -8 & 14 & -4 \\ -11 & 17 & -4 \end{bmatrix} \xrightarrow[R3-R1]{R2-R1} \begin{bmatrix} -5 & 11 & -4 \\ -3 & 3 & 0 \\ -6 & 6 & 0 \end{bmatrix} \xrightarrow[-1/3 2R2]{R3-2R2} \begin{bmatrix} -5 & 11 & -4 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then $x_1 = x_2$, and $4x_3 = -5x_1 + 11x_2 = 6x_1$. With $x_3 = t$, we get the eigenvectors for $\lambda = -1$ to be

$$t \begin{bmatrix} 2/3 \\ 2/3 \\ 1 \end{bmatrix}.$$

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} -8 & 11 & -4 \\ -8 & 11 & -4 \\ -11 & 17 & -7 \end{bmatrix} \xrightarrow[R3-11/8R1]{R2-R1} \begin{bmatrix} -8 & 11 & -4 \\ 0 & 0 & 0 \\ 0 & 15/8 & -3/2 \end{bmatrix} \xrightarrow{8R3} \begin{bmatrix} -8 & 11 & -4 \\ 0 & 0 & 0 \\ 0 & 15 & -12 \end{bmatrix}$$

If $x_3 = t$, then $15x_2 = 12t$, or $x_2 = 4t/5$. From the first equation, $-8x_1 + 44t/5 - 4t = 0$, so $x_1 = 24t/40 = 3t/5$. So the eigenvectors for $\lambda = 2$ are

$$t \begin{bmatrix} 3/5 \\ 4/5 \\ 1 \end{bmatrix}.$$

For $\lambda = 1$,

$$A - I = \begin{bmatrix} -7 & 11 & -4 \\ -8 & 12 & -4 \\ -11 & 17 & -6 \end{bmatrix} \xrightarrow[R2+8R1, R3+11R1]{R1-R2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & -4 \\ 0 & 6 & -6 \end{bmatrix}.$$

Thus $x_1 = x_2 = x_3$. The eigenvectors for $\lambda = 1$ are

$$t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Now we can diagonalize A: we form

$$P = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}.$$

Then

$$P^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix},$$

and

$$A = PDP^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

So

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} (-1)^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 1^{10} \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -1024 & 1024 & 0 \\ 2 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -3068 & 3069 & 0 \\ -4092 & 4093 & 0 \\ -5115 & 5115 & 1 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 9 & -1 & -8 \\ 3 & 5 & -8 \\ 0 & 0 & 6 \end{bmatrix}.$$

Characteristic polynomial: along the third row,

$$\begin{vmatrix} 9-\lambda & -1 & -8 \\ 3 & 5-\lambda & -8 \\ 0 & 0 & 6-\lambda \end{vmatrix} = (6-\lambda)\left[(9-\lambda)(5-\lambda)+3\right] = (6-\lambda)(\lambda^2-14\lambda+48)$$

$$= (6-\lambda)(\lambda-6)(\lambda-8).$$

So the eigenvalues are $6, 6, 8$.

Eigenvectors: for $\lambda = 6$, we need to solve the homogeneous system with matrix

$$\begin{bmatrix} 9-6 & -1 & -8 \\ 3 & 5-6 & -8 \\ 0 & 0 & 6-6 \end{bmatrix} = \begin{bmatrix} 3 & -1 & -8 \\ 3 & -1 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

Row reduction gives us immediately

$$\begin{bmatrix} 3 & -1 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So we take x_2 and x_3 as parameters, say $x_2 = r$, $x_3 = s$, and $x_1 = \frac{1}{3}(r+8s)$. That is, eigenvectors for the eigenvalue 6are of the form

$$\begin{bmatrix} \frac{1}{3}(r+8s) \\ r \\ s \end{bmatrix} = r\begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 8/3 \\ 0 \\ 1 \end{bmatrix}.$$

By replacing the coefficients, we may rewrite the eigenvectors as

$$r\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + s\begin{bmatrix} 8 \\ 0 \\ 3 \end{bmatrix}.$$

For the eigenvalue 8, the homogeneous system becomes

$$\begin{bmatrix} 9-8 & -1 & -8 \\ 3 & 5-8 & -8 \\ 0 & 0 & 6-8 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -8 \\ 3 & -3 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

We can perform row reduction as follows:

$$\begin{bmatrix} 1 & -1 & -8 \\ 3 & -3 & -8 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -8 \\ 0 & 0 & 16 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_3 = 0$, and making $x_2 = t$, we have $x_1 = t$. So the eigenvectors are of the form

$$\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

To diagonalize A, we form P using the basic eigenvectors,

$$P = \begin{bmatrix} 1 & 8 & 1 \\ 3 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

Then

$$P^{-1} = \begin{bmatrix} -1/2 & 1/2 & 4/3 \\ 0 & 0 & 1/3 \\ 3/2 & -1/2 & -4 \end{bmatrix},$$

and

$$A = PDP^{-1} = \begin{bmatrix} 1 & 8 & 1 \\ 3 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 & 4/3 \\ 0 & 0 & 1/3 \\ 3/2 & -1/2 & -4 \end{bmatrix}.$$

Thus

$$A^{10} = PD^{10}P^{-1} = \begin{bmatrix} 1 & 8 & 1 \\ 3 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 6^{10} & 0 & 0 \\ 0 & 6^{10} & 0 \\ 0 & 0 & 8^{10} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 & 4/3 \\ 0 & 0 & 1/3 \\ 3/2 & -1/2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 8 & 1 \\ 3 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 6^{10} & 0 & 0 \\ 0 & 6^{10} & 0 \\ 0 & 0 & 8^{10} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 & 4/3 \\ 0 & 0 & 1/3 \\ 3/2 & -1/2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1580379648 & -506637824 & -4053102592 \\ 1519913472 & -446171648 & -4053102592 \\ 0 & 0 & 60466176 \end{bmatrix}.$$