Math 122-002 201730
Practice Assignment # 11 – Answers

1. Compute $\vec{u} \times \vec{v}$, $\vec{v} \times \vec{u}$, and verify that one is the negative of the other.

   (a) $\vec{u} = -5\check{\mathbf{i}} + 6\check{\mathbf{j}} + \check{\mathbf{k}}$, $\vec{v} = -\check{\mathbf{i}} + 2\check{\mathbf{j}} + \check{\mathbf{k}}$
   Answer.

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \check{\mathbf{i}} & \check{\mathbf{j}} & \check{\mathbf{k}} \\ -5 & 6 & 1 \\ -1 & 2 & 1 \end{vmatrix} \\
&= (6-2)\check{\mathbf{i}} + (-1+5)\check{\mathbf{j}} + (-10+6)\check{\mathbf{k}} \\
&= 4\check{\mathbf{i}} + 4\check{\mathbf{j}} - 4\check{\mathbf{k}}. \\
\vec{v} \times \vec{u} &= \begin{vmatrix} \check{\mathbf{i}} & \check{\mathbf{j}} & \check{\mathbf{k}} \\ -1 & 2 & 1 \\ -5 & 6 & 1 \end{vmatrix} \\
&= (2-6)\check{\mathbf{i}} + (-5+1)\check{\mathbf{j}} + (-6+10)\check{\mathbf{k}} \\
&= -4\check{\mathbf{i}} - 4\check{\mathbf{j}} + 4\check{\mathbf{k}}.
\end{aligned}$$

   (b) $\vec{u} = 5\check{\mathbf{i}} + 7\check{\mathbf{j}} + 4\check{\mathbf{k}}$, $\vec{v} = 20\check{\mathbf{i}} - 6\check{\mathbf{k}}$.
   Answer.

$$\begin{aligned}
\vec{u} \times \vec{v} &= \begin{vmatrix} \check{\mathbf{i}} & \check{\mathbf{j}} & \check{\mathbf{k}} \\ 5 & 7 & 4 \\ 20 & 0 & -6 \end{vmatrix} \\
&= -42\check{\mathbf{i}} + (80+30)\check{\mathbf{j}} + (0-140)\check{\mathbf{k}} \\
&= -42\check{\mathbf{i}} + 110\check{\mathbf{j}} - 140\check{\mathbf{k}}. \\
\vec{v} \times \vec{u} &= \begin{vmatrix} \check{\mathbf{i}} & \check{\mathbf{j}} & \check{\mathbf{k}} \\ 20 & 0 & -6 \\ 5 & 7 & 4 \end{vmatrix} \\
&= 42\check{\mathbf{i}} + (-30-80)\check{\mathbf{j}} + (140-0)\check{\mathbf{k}} \\
&= 42\check{\mathbf{i}} - 110\check{\mathbf{j}} + 140\check{\mathbf{k}}.
\end{aligned}$$

2. In each case find two vectors, both normal to the given plane.

(a) $8x - y + z = 12$

Answer. Since there is only one direction orthogonal to the plane, the two vectors will be colinear. $8x - y + z = 12$ has normal vector $8\check{\mathbf{i}} - \check{\mathbf{j}} + \check{\mathbf{k}}$. To find another vector we multiply by a scalar, let's say 3: $24\check{\mathbf{i}} - 3\check{\mathbf{j}} + 3\check{\mathbf{k}}$.

(b) $4x + 6y + 4z = -5$.

Answer. Again, a normal vector is given by $4\check{\mathbf{i}} + 6\check{\mathbf{j}} + 4\check{\mathbf{k}}$. Another multiple of this vector will do. Say we take its opposite, $-4\check{\mathbf{i}} - 6\check{\mathbf{j}} - 4\check{\mathbf{k}}$.

3. Find the equation of the plane containing the given point and having the given vector as normal vector

(a) $(-1, -1, -2)$, $3\check{\mathbf{i}} - \check{\mathbf{j}} + 4\check{\mathbf{k}}$

Answer. The equation of the plane through A and orthogonal to $\vec{v}$ is $(X - A) \cdot \vec{v} = 0$. So

$$\big[(x - (-1))\check{\mathbf{i}} + (y - (-1)\check{\mathbf{j}} + (z - (-2))\check{\mathbf{k}}\big] \cdot [3\check{\mathbf{i}} - \check{\mathbf{j}} + 4\check{\mathbf{k}}] = 0.$$

Calculating,

$$3(x + 1) - (y + 1) + 4(z + 2) = 0,$$

or

$$3x - y + 4z = -10.$$

(b) $(2, -3, 4)$, $8\check{\mathbf{i}} - 6\check{\mathbf{j}} + 4\check{\mathbf{k}}$

Answer.

$$\big[(x - 2)\check{\mathbf{i}} + (y - (-3))\check{\mathbf{j}} + (z - 4)\check{\mathbf{k}}\big] \cdot [8\check{\mathbf{i}} - 6\check{\mathbf{j}} + 4\check{\mathbf{k}}] = 0.$$

Calculating,

$$8(x - 2) - 6(y + 3) + 4(z - 4) = 0,$$

or

$$8x - 6y + 4z = 50.$$

(c) $(-2, -1, 0)$, $4\check{\mathbf{i}} + 2\check{\mathbf{j}} + \check{\mathbf{k}}$

Answer.

$$\big[(x + 2)\check{\mathbf{i}} + (y + 1)\check{\mathbf{j}} + z\check{\mathbf{k}}\big] \cdot [4\check{\mathbf{i}} + 2\check{\mathbf{j}} + \check{\mathbf{k}}] = 0.$$

Calculating,

$$4(x + 2) + 2(y + 1) + z = 0,$$

or

$$4x + 2y + z = -10.$$

4. Find the equation of the plane containing the points $(0, 1, 2)$, $(-1, 2, 3)$, $(1, 1, 4)$.

Answer. Let us choose one of the points, say $(0, 1, 2)$; now consider the vectors from this point to each of the other two. That is, let

$$\vec{u} = (-1 - 0)\check{\mathbf{i}} + (2 - 1)\check{\mathbf{j}} + (3 - 2)\check{\mathbf{k}} = -\check{\mathbf{i}} + \check{\mathbf{j}} + \check{\mathbf{k}},$$

$$\vec{v} = (1-0)\check{\mathbf{i}} + (1-1)\check{\mathbf{j}} + (4-2)\check{\mathbf{k}} = \check{\mathbf{i}} + 2\check{\mathbf{k}}.$$

By doing the cross product of these two vectors, we obtain a vector normal to the plane:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \check{\mathbf{i}} & \check{\mathbf{j}} & \check{\mathbf{k}} \\ -1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 2\check{\mathbf{i}} + 3\check{\mathbf{j}} - \check{\mathbf{k}}.$$

Now we have a vector normal to the plane, so we can write the equation of the plane using one of the points:

$$2(x-0) + 3(y-1) - (z-2) = 0,$$

or

$$2x + 3y - z = 1$$

5. Find the equation of the line in $\mathbb{R}^2$ that passes through the point $(2,3)$ and is perpendicular to the line $5x + y = 2$.

   Answer. The line $5x + y = 2$ passes through the points $A = (0,2)$ and $B = (1,-3)$, so a directing vector will $A - B = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$; so the line can be can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$

   So we need a vector perpendicular to $\begin{bmatrix} -1 \\ 5 \end{bmatrix}$; if $\begin{bmatrix} a \\ b \end{bmatrix}$ is such a vector, since the dot product should be zero we need

$$-a + 5b = 0$$

   We can take, for instance, $a = 5$, $b = 1$. So the line

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

   satisfies the requirements.

6. Find the equation of the line in $\mathbb{R}^3$ that passes through the point $(1,1,-2)$ and is perpendicular to the plane $3x + 5y - 2z = 3$.

   Answer. We read from the equation of the plane that $\begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$ is perpendicular to the plane; so the line should go along this vector. Then we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}.$$

7. The intersection any two planes that are not parallel is a line. Find the equation of the line that is the intersection of the planes $x + y + z = 1$ and $x - y - z = 2$.

Answer. The intersection of the two planes are the points (x, y, z) satisfying

$$\begin{cases} x + y + z = 1 \\ x - y - z = 2 \end{cases}$$

We solve by row reduction:

$$\begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 1 & -1 & -1 & | & 2 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & -2 & -2 & | & 1 \end{bmatrix} \xrightarrow{-R2/2} \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 1 & 1 & | & -1/2 \end{bmatrix}$$

$\xrightarrow{R1-R2} \begin{bmatrix} 1 & 0 & 0 & | & 3/2 \\ 0 & 1 & 1 & | & -1/2 \end{bmatrix}$ So $x = 3/2$, $y + z = -1/2$. Taking $z = t$ as a parameter, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 \\ -1/2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$