University of Regina
Department of Mathematics and Statistics
Math 122-002 201730
Midterm 2 – Answers

1. Use the inversion algorithm to find the inverse where possible. If the matrix is invertible, write the inverse as a product of elementary matrices.

(a) $\begin{bmatrix} 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & -2 & 3 \end{bmatrix}$

Answer.

(a) The matrix A has a row of zeroes, so it is not invertible.

(b) We have

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1\leftrightarrow R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R2\leftrightarrow R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right] \xrightarrow{(1/2)R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R2-3R3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & -3/2 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right] \xrightarrow{(-1/2)R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3/4 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 3/4 & 0 & -1/2 \\ 1/2 & 0 & 0 \end{bmatrix}$$

Collecting the elementary matrices,

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Find the value of c for which A has no inverse

$$A = \begin{bmatrix} 1 & 3 \\ c & 7 \end{bmatrix}$$

Answer. The determinant of A is $|A| = 7 - 3c$. For A to have no inverse, we need this to be zero. That is, $c = 7/3$.

3. Calculate the determinant using row/column reduction.

$$A = \begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix}.$$

Answer. Doing $R3 - R2$, $R4 - 2R2$, $R1 - R3$, $R3 - 2R4$, and then calculating along the first column,

$$\begin{vmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -1 \\ 2 & 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 & 5 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 2 & 5 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & -2 & -1 \\ 0 & 1 & 1 & 0 \end{vmatrix} =$$
$$- \begin{vmatrix} 0 & 2 & 5 \\ 0 & -2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -(2 \times (-1) - (-2) \times 5) = -8.$$

4. Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Diagonalize A and use this information to calculate A^{20} *(Helpful fact:* $2^{20} = 1048576$*)*

Answer. Eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & -1 \\ 1 & 1-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda).$$

So the eigenvalues are $1, 1, 2$.

Eigenvectors. For $\lambda = 1$, we solve the homogeneous system with matrix $A - I$. That is',

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This leads to $x = z$, and we are free to choose y. Setting $y = s$, $z = t$, we get eigenvectors

$$\begin{bmatrix} t \\ s \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda = 2$, we row-reduce $A - 2I$, that is

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $x = y$, and $z = 0$. Letting $y = r$, we get the eigenvectors

$$r \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

We form

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

We have

$$P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Then

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}^{20} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{20} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2^{20} \\ 0 & 1 & 2^{20} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2^{20} & 0 & 1 - 2^{20} \\ 2^{20} - 1 & 1 & 1 - 2^{20} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1048576 & 0 & -1048575 \\ 1048575 & 1 & -1048575 \\ 0 & 0 & 1 \end{bmatrix}$$