University of Regina
Department of Mathematics and Statistics
Math 122-002 201730
Midterm 2 – Answers

1. Use the inversion algorithm to find the inverse where possible. If the matrix is invertible, write the inverse as a product of elementary matrices.

(a) $\begin{bmatrix} 1 & 2 & 5 & 8 \\ 7 & -1 & 6 & 6 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix}$

Answer.

(a) Here the matrix A has a row of zeroes, so it is not invertible.

(b) We perform row reduction:

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R1 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} -2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|ccc} -2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{(1/2)R2} \left[\begin{array}{ccc|ccc} -2 & 3 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R1-3R2} \left[\begin{array}{ccc|ccc} -2 & 0 & 0 & -3/2 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{(-1/2)R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

So A is invertible, $A^{-1} = \begin{bmatrix} 3/4 & 0 & -1/2 \\ 1/2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, and collecting the elementary matrices we have

$$A^{-1} = \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

2. Find the value of b for which A has no inverse

$$A = \begin{bmatrix} 1 & b \\ 2 & 9 \end{bmatrix}$$

Answer. The determinant of A is $9 - 2b$, which will only be zero when $b = 9/2$.

3. Calculate the determinant using row/column reduction.

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 4 \end{bmatrix}.$$

Answer. Doing $C4 - 2C3$, $R4 - 2R3$, $C3 - C1$, $C2 - C3$, and calculating along the first row both times,

$$\begin{vmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 1 \\ 2 & 1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & -1 \\ 2 & 1 & 3 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & -1 \\ 0 & -3 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & -3 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 \\ 0 & -4 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 0 & -4 & 0 \end{vmatrix} = 2 \times 0 - (-4)(-1) = -4.$$

4. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Diagonalize A and use this information to calculate A^{19} *(Helpful fact:* $2^{19} = 524288$*)*

Answer. We first find the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda).$$

So the eigenvalues are $1, 1, 2$.

Now the eigenvectors. For $\lambda = 1$, we solve the homogeneous system with matrix $A - \lambda$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so $z = 0$ and $x = s$, $y = t$ are parameters. The eigenvectors are of the form

$$\begin{bmatrix} s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda = 2$, we solve

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $x = z$, $y = 0$. We take $z = t$, and so the eigenvectors are of the form

$$\begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

So we form

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This is an elementary matrix, so we see directly that the inverse is

$$P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$A^{19} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{19} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 524288 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 524288 \\ 0 & 1 & 0 \\ 0 & 0 & 524288 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 524287 \\ 0 & 1 & 0 \\ 0 & 0 & 524288 \end{bmatrix}$$