University of Regina
Department of Mathematics and Statistics
Math 122-002 201730
Quiz # 2 – Answers

1. In each case, find three 2×2 matrices A such that

   (a) $A^2 = I$;

   (b) $A^2 = A$.

   Answer. This was 2.3.8.b in the textbook.

   (a) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

   $$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix}$$

   The equality $A^2 = I$ generates then four equations,

   $$\begin{aligned} a^2 + bc &= 1 \\ ab + bd &= 0 \\ ac + dc &= 0 \\ bc + d^2 &= 1 \end{aligned}$$

   From the second equation, $b(a + d) = 0$, and from the third one $c(a + d) = 0$. So on possible choice is $b = c = 0$, and the other equations become $a^2 = d^2 = 1$. This gives us four matrices

   $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

   Another option is to go with $a + d = 0$ and solve $bc = 1 - a^2$. For instance, $b = 1$, $c = 1 - a^2$. So we get, not three, but infinitely many matrices

   $$\begin{bmatrix} a & 1 \\ 1 - a^2 & -a \end{bmatrix},$$

   where a can be any real number. We can take three matrices out of this by taking values for a, say $0, 1, 27$:

   $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 27 & 1 \\ -728 & -27 \end{bmatrix}.$$

   There are still more possible solutions, by solving the equations for $b \neq 1$.

(b) Now the four equations become

$$\begin{aligned} a^2 + bc &= a \\ ab + bd &= b \\ ac + dc &= c \\ bc + d^2 &= d \end{aligned}$$

If $b = 0$, the second equation is automatically satisfied. This forces $a^2 = a$, $d^2 = d$, so a and d can only be 0 or 1. And for the third equation we could also take $c = 0$. Then

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are many more possibilities when we allow $b \neq 0$. In that case,

$$b = b(a + d),$$

so $a + d = 1$. This third equation is automatically satisfied, so we are left with

$$a^2 + bc = a, \quad (1 - a)^2 + bc = 1 - a.$$

It is easy to check that the second equation is the same as the first one. So we only have $bc = a - a^2$. If $0 \leq a \leq 1$ and $c = b$, we get infinitely many matrices of the form

$$\begin{bmatrix} a & \sqrt{a - a^2} \\ \sqrt{a - a^2} & 1 - a \end{bmatrix}, \quad 0 \leq a \leq 1.$$

More generally, we get

$$\begin{bmatrix} a & b \\ (a - a^2)/b & 1 - a \end{bmatrix},$$

where we are free to choose a and b. We can get as many concrete examples as we want. For example, with $a = 0$, $b = 1$; $a = 1/2$, $b/1/2$; and $a = 30$, $b = -5$, we get respectively

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \quad \begin{bmatrix} 30 & -5 \\ 174 & -29 \end{bmatrix}.$$

If instead of $b \neq 0$ we force $c \neq 0$, we can reason in an analog fashion to get

$$\begin{bmatrix} a & (a - a^2)/c \\ c & 1 - a \end{bmatrix},$$

and here we are free to choose a and c, and again we can get as many examples as we want.