Math 217-001 201810
Practice Assignment # 1 – Answers

1. For each of the following DE, decide whether they are linear and/or separable and state their order. ***Warning:*** *some algebra may be needed before deciding!*

   (a) $y'y = x$

   (b) $y'/y = x$

   (c) $y'' - e^x y' + 2e^x y = 0$

   Answer.

   (a) First order separable, but not linear.

   (b) We can write the equation as $y' - xy = 0$, so first order, linear, and separable.

   (c) Second order, linear.

2. (a) Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$.

   (b) Is the equation $y' = 1 + y^2$ separable?

   (c) Solve the initial value problem $y' = 1 + y^2$, $y(0) = 1$. What is the interval of definition for the solution?

   Answer.

   (a) If $y(x) = \tan(x + c)$, then $y' = 1/\cos^2(x + c)$, and

$$1 + y^2 = 1 + \tan^2(x + c) = 1 + \frac{\sin^2(x + c)}{\cos^2(x + c)} = \frac{\cos^2(x + c) + \sin^2(x + c)}{\cos^2(x + c)}$$
$$= \frac{1}{\cos^2(x + c)}.$$

   (b) Yes. As $1 + y^2 > 0$ for all x, the equation can be written as $y'/(1 + y^2) = 1$, separable.

(c) We already have a family of solutions, but it is also the only one: taking anti-derivatives on $y/(1+y^2)$,

$$\arctan y = x + c,$$

so $y(x) = \tan(x+c)$. From the initial condition, $1 = y(0) = \tan(c)$. We have infinitely many choices for c (because the tangent function is periodic), but the most canonical choice would be $c = \pi/4$. Around 0, the tangent function is defined on the interval $(-\pi/2, \pi/2)$, so our solution requires $-\pi/2 < x + \pi/4 < \pi/2$, i.e.

$$y(x) = \tan(x + \pi/4), \quad x \in (-3\pi/4, \pi/4).$$

3. Solve the IVP. *Recall that a solution to an IVP always includes the interval of definition*

(a) $y' = y + 1$, $y(0) = -3$

(b) $\dfrac{dy}{dx} = x^3\sqrt{1-y^2}$, $y(0) = 1$

(c) $z' = \dfrac{1-z^2}{1-x^2}$, $z(2) = 2$.

Answer.

(a) Separable equation. We have $y'/(1+y) = 1$, and after taking antiderivatives we have

$$\log(1+y) = x + c.$$

Applying the exponential to both sides of the equality, $1 + y(x) = e^{x+c} = e^x\, e^c$. After renaming e^c as d, we get

$$y(x) = de^x - 1.$$

The initial condition gives $-3 = y(0) = d - 1$, so $d = -2$. This is defined everywhere and satisfies the equation everywhere, so the solution is

$$y(x) = -2e^x - 1, \quad x \in (-\infty, \infty).$$

(b) This is separable equation. We write $y'/\sqrt{1-y^2} = x^3$. Taking antiderivatives on both sides,

$$\arcsin y = \frac{x^4}{4} + c,$$

so $y(x) = \sin\left(\frac{x^4}{4} + c\right)$. From the initial condition, $1 = y(0) = \sin(c)$. We can choose for example $c = \pi/2$. For the original equation to be satisfied, we need the relation

$$\cos\left(\frac{x^4}{4} + \frac{\pi}{2}\right) = \sqrt{1 - \sin\left(\frac{x^4}{4} + \frac{\pi}{2}\right)}.$$

This requires the cosine to be positive. The biggest interval that contains 0 and where the cosine is positive is $[-\pi/2, \pi/2]$. So the solution is

$$y(x) = \sin\left(\frac{x^4}{4} + \frac{\pi}{2}\right), \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

(c) Separating the variables,

$$\frac{z'}{1-z^2} = \frac{1}{1-x^2}.$$

To find the anti-derivatives we need to use partial fractions. If we write (using that $1 - z^2 = (1-z)(1+z)$)

$$\frac{1}{1-z^2} = \frac{A}{1-z} + \frac{B}{1+z} = \frac{A - Az + B - Bz}{1-z^2} = \frac{A + B + (B-A)z}{1-z^2},$$

we deduce that $A + B = 1$, $B - A = 0$. So $A = B = 1/2$. So now we have

$$\frac{1}{2}\left(\frac{1}{1+z} + \frac{1}{1-z}\right) z' = \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right).$$

After cancelling the 1/2 on both sides and taking antiderivatives,

$$\log(1+z) - \log(1-z) = \log(1+x) - \log(1-x) + c$$

for an arbitrary real number c. Taking exponential on both sides,

$$\frac{1+z}{1-z} = e^c \, \frac{1+x}{1-x}$$

Now we rename e^c as d, and solve for z:

$$z(x) = \frac{d(1+x)+x-1}{d(1+x)-x+1}$$

The initial condition gives $2 = z(2) = (3d+1)/(3d-1)$. Then $d = 1$. So $z(x) = x$ (with $d = 1$, the numerator is $2x$, and the denominator is 2). The equation does not allow $x = \pm 1$, because the denominator becomes 0. As our interval of definition has to contain $x = 2$, we take $(1, \infty)$. So the solution is

$$z(x) = x, \quad x \in (1, \infty).$$

4. A large mixing tank contains 300 litres of water, in which 50 kg of salt have been dissolved. Brine solution, with a salt concentration of 0.1 kg/litre, is pumped into the tank at a rate of 3 litres per minute. The mixed solution is pumped out at a rate of 2 litres per minute. Determine the amount of salt $A(t)$ in the tank at any given time $t \geq 0$.

   Answer. We know that $A(0) = 50$ kg. And the rate of change of $A(t)$ is

$$A'(t) = 0.1 \frac{\text{kg}}{\text{lit}}\, 3 \frac{\text{lit}}{\text{min}} - \frac{A(t)}{(300+t)\ \text{lit}}\, 2 \frac{\text{lit}}{\text{min}} = (0.3 - 2A(t)/(300+t))\, \frac{\text{kg}}{\text{min}}.$$

   So we need to solve the DE $A' = 0.3 - 2A(t)/(300+t)$, or $A' + \frac{2A(t)}{(300+t)} = 0.3$. This is a linear first order equation. The integrating factor is

$$\mu(t) = e^{\int \frac{2}{300+t}\, dt} = e^{2 \log(300+t)} = (300+t)^2.$$

   Thus our equation becomes

$$[(300+t)^2 \, A(t)]' = 0.3 \, (300+t)^2,$$

   from where we obtain

$$A(t) = 0.1 \, (300+t) + \frac{c}{(300+t)^2}.$$

   From $A(0) = 50$, we get $c = 1800000$. So

$$A(t) = 30 + \frac{t}{10} + \frac{1800000}{(300+t)^2}.$$

5. Tired of studying for Math 217, you decide to drink a can of cold pop. When you go to the kitchen, you find your roommate has drank all the cans from the fridge, and only warm cans are available, sitting at 25 degrees. You put a can in the freezer (which is at -18 degrees), and after 5 minutes its temperature is now 17 degrees. How long should you wait to get the temperature of the can to the ideal 3 degrees?

   Answer. The relevant equation here is Newton's Law of Cooling: $T'(t) = k(T(t) - T_{\text{amb}})$, where T_{amb} is the ambient temperature. Solving the separable equation (we did it in class),

   $$T(t) = T_{\text{amb}} + d\, e^{kt}.$$

   We know that $T_{\text{amb}} = -18$ and that $T(0) = 25$, so $25 = T(0) = -18 + d$, and $d = 25 + 18 = 43$. Thus $T(t) = -18 + 43e^{kt}$. The problem tells us that $T(5) = 17$, so

   $$17 = -18 + 43e^{5k}.$$

   Solving for k, we obtain

   $$k = \frac{1}{5} \log \frac{35}{43} = -0.04117$$

   (approximately). We want to solve for t in $3 = T(t)$, so

   $$3 = -18 + 43\, e^{-0.04117t},$$

   or

   $$t = -\frac{\log \frac{18+3}{43}}{0.04117} = 17.41$$

   (approximately). That is, you have to wait 17 minutes and 24 seconds ($.41 \times 60 = 24$).