Math 217-001 201810
Practice Assignment # 3 – Answers

1. Solve, giving the largest possible interval in which a solution is defined.

   (a) $y' = \dfrac{\sin\sqrt{x}}{\sqrt{y}}$, $y(\pi^2) = 1$.

   (b) $t(t+1)\,\dfrac{dv}{dt} + t\,v = 1$, $v(e) = 1$.

   (c) $y' + (\tan x)\,y = \cos^2 x$, $y(0) = -1$.

Answer.

(a) This is the separable equation $y^{1/2}\,y' = \sin\sqrt{x}$. The antiderivatives of the two sides should differ by a constant. Using substitution and parts,

$$\begin{aligned}\frac{2}{3}\,y^{3/2} &= c + \int \sin\sqrt{x}\,dx = c + \int 2u\sin u\,du = c - 2u\cos u + \int 2\cos u\,du\\ &= c - 2\sqrt{x}\,\cos\sqrt{x} + 2\sin\sqrt{x}.\end{aligned}$$

From the initial condition:

$$\frac{2}{3} = \frac{2}{3}y(\pi^2)^{3/2} = c - 2\pi\cos\pi + 2\sin\pi = c + 2\pi,$$

so $c = 2/3 - 2\pi$. The solution is then

$$y(x) = (1 - 3\pi - 3\sqrt{x}\,\cos\sqrt{x} + 3\sin\sqrt{x})^{2/3},$$

We need $x > 0$ and $y > 0$ (because of the square roots). It is not easy to determine explicitly where we have $y > 0$. A look at the graph suggests approximately the interval $[7.02266, 12.857]$ (recall that it has to contain $\pi^2 \simeq 9.86$).

(b) This is a First Order Linear DE. If we look for the integrating factor, we first need to write

$$v' + \frac{1}{t+1}\,v = \frac{1}{t(t+1)}.$$

Then $\mu(t) = e^{\int 1/(t+1)\,dt} = e^{\log(t+1)} = t+1$. Now we know that our equation looks like

$$[(t+1)v]' = \frac{1}{t},$$

or $(t+1)v = c + \log t$. From the initial condition, $c + \log e = (e+1)v(e) = e+1$, so $c = e$. As for the interval of validity, we need $t \neq 0$, $t \neq -1$; as $e > 0$, we can take $(0, \infty)$. The solution is then

$$v(t) = \frac{e + \log t}{t+1}, \quad t > 0.$$

(c) Again this is first order linear. We look for the integrating factor: an antiderivative of $\tan x$ is $-\log\cos x$, so we can take our integrating factor to be $\mu(x) = e^{-\log\cos x} = 1/\cos x$. After multiplying both sides by μ, our equation looks like

$$\frac{1}{\cos x}\, y' + \frac{\sin x}{\cos^2 x}\, y = \cos x,$$

or $(y/\cos x)' = \cos x$. Looking at the antiderivatives, $y/\cos x = c + \sin x$, or

$$y(x) = c\,\cos x + \sin x\,\cos x.$$

From the initial condition, $-1 = y(0) = c$. We need $\cos x \neq 0$ (for the tangent to be defined). The biggest such interval that contains 0 is $(-\pi/2, \pi/2)$. So the solution is

$$y(x) = \sin x\,\cos x - \cos x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

2. Find a continuous solution

$$\begin{cases} (1+x^2)y' + 2x\,y = f(x), \\ y(0) = 0 \end{cases}$$

where

$$f(x) = \begin{cases} x, & x \in [0,1] \\ -x, & x \in (1,\infty) \end{cases}$$

Answer. We work first on $[0,1]$. In that interval, our equation looks like

$$(1+x^2)y' + 2x\,y = x.$$

We could look for the integrating factor, or just notice that the left-hand-side is $[(1+x^2)y]'$. So, after taking antiderivatives,

$$(1+x^2)y = \frac{x^2}{2} + c.$$

The initial condition gives us $c = \frac{0^2}{2} = (1+0^2)y(0) = y(0) = 0$. So, on $[0,1]$, $y(x) = x^2/(2+2x^2)$.

On $(1,\infty)$, our equation becomes $[(1+x^2)y]' = -x$. So

$$y(x) = \frac{c - x^2/2}{1+x^2}$$

(careful: this is a different c!). As we want y to be continuous, we need the two functions we found to agree at $x=1$. That is,

$$\frac{1}{2+2} = \frac{c-1/2}{1+1}.$$

That is, $c = 1$. As $1+x^2 > 0$ for all x, there are no domain restrictions. So the solution is

$$y(x) = \begin{cases} \dfrac{x^2}{2+2x^2}, & x \in [0,1] \\ \\ \dfrac{2-x^2}{2+2x^2}, & x \in (1,\infty) \end{cases}, \qquad x \in (-\infty, \infty).$$

3. Solve

   (a) $\begin{cases} y' = -\dfrac{20x^3y - 6x}{5x^4+4y}, \\ y(0) = 1 \end{cases}$

   (b) $y(x+y+1) + (x+2y)y' = 0$, $y(0) = 0$.

   (c) $y^2(x^2+1) + (x^3y + 3xy)\,y' = 0$.

   Answer.

(a) This looks like an exact first order DE. We have $M(x,y) = 20x^3y - 6x$, $N(x,y) = 5x^4 + 4y$. We check for exactness:

$$\frac{\partial M}{\partial y} = 20x^3, \quad \frac{\partial N}{\partial x} = 20x^3.$$

They are equal, so the equation is exact. Now we work to find the potential function. From

$$\frac{\partial \phi}{\partial x} = M(x,y) = 20x^3y - 6x,$$

we get $\phi(x,y) = 5x^4y - 3x^2 + h(y)$. Differentiating with respect to y and equating with N we get

$$5x^4 + h'(y) = \frac{\partial \phi}{\partial y} = N(x,y) = 5x^4 + 4y.$$

So $h'(y) = 4y$ and then $h(y) = 2y^2 + c$ for some constant c. Thus, $\phi(x,y) = 5x^4y - 3x^2 + 2y^2 + c$. A solution to our equation satisfies $0 = \phi(x,y) = 5x^4y - 3x^2 + 2y^2 + c$. From $y(0) = 1$, we get $0 = 2 + c$, so $c = -2$. A solution to our IVP satisfies $5x^4y - 3x^2 + 2y^2 - 2 = 0$. Solving the quadratic on y, $y(x)$ is necessarily of the form

$$y(x) = \frac{-5x^4 \pm \sqrt{25x^8 + 8(3x^2+2)}}{4}.$$

As we need y to be positive around 0 (because $y(0) = 1$), only the solution with $+$ solves our problem:

$$y(x) = \frac{-5x^4 + \sqrt{25x^8 + 8(3x^2+2)}}{4}, \quad -\infty < x < \infty.$$

(b) We can quickly check that this equation is not exact. But we can try to make it exact by using an appropriate integrating factor. Such an integrating factor would have to be

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx},$$

provided that $(M_y - N_x)/N$ does not depend on y. We have

$$\frac{M_y - N_x}{N} = \frac{x + 2y + 1 - 1}{x + 2y} = 1.$$

So $\mu(x) = e^x$. After multiplying by e^x, our equation now looks like
$$e^x y(x+y+1) + e^x(x+2y)y' = 0.$$
This is indeed exact:
$$\frac{\partial}{\partial y}(xye^x + y^2e^x + ye^x) = xe^x + 2ye^x + e^x = (x+2y+1)e^x,$$
$$\frac{\partial}{\partial x}(xe^x + 2ye^x) = e^x + xe^x + 2ye^x = (x+2y+1)e^x.$$
We now find the potential function: from $\frac{\partial \phi}{\partial x} = xye^x + y^2e^x + ye^x$ we get
$$\phi(x,y) = (x-1)ye^x + y^2e^x + ye^x + h(y) = (xy+y^2)e^x + h(y).$$
Now from $\frac{\partial \phi}{\partial y} = e^x(x+2y)$ we get
$$(x+2y)e^x + h'(y) = e^x(x+2y).$$
So $h'(y) = 0$, and $h(y) = c$ for some constant c. Our potential function is them $\phi(x,y) = (xy+y^2)e^x + c$, and any solution to the equation has to satisfy
$$(xy+y^2)e^x + c = 0.$$
This is a quadratic equation; with conditions on the positivity of y it would be possible to derive explicit solutions. In this cse we have more information: from the initial condition, $y(0) = 0$, we obtain $c = 0$.
$$e^x\, y^2 + xe^x\, y = 0,$$
so whenever $y \neq 0$
$$y = -xe^x e^{-x} = -x.$$

(c) A quick check shows that this is not exact. If we try
$$\frac{M_y - N_x}{N} = \frac{2y(x^2+1) - 3x^2y - 3y}{x^3y + 3xy} = \frac{2yx^2 - y - 3x^2y}{x^3y + 3xy},$$

with no obvious simplification. On the other hand,

$$\frac{N_x - M_y}{M} = \frac{x^2y + y}{y^2(x^2+1)} = \frac{1}{y}.$$

So $\mu'/\mu = 1/y$, which means $mu(y) = y$. So we multiply our equation by y, to obtain the exact equation

$$y^3(x^2+1) + (x^3y^2 + 3xy^2)\, y' = 0$$

Now we look for the potential function. We have $\varphi_x = M$, so

$$\varphi(x,y) = y^3\left(\frac{x^3}{3} + x\right) + g(y).$$

Looking now at N, we have

$$x^3y^2 + 3xy^2 = \varphi_y = 3y^2\left(\frac{x^3}{3} + x\right) + g'(y).$$

We see that $g'(y) = 0$, so g is constant. The implicit equation for the solution to our problem is

$$y^3\left(\frac{x^3}{3} + x\right) = c.$$

This is easy to solve:

$$y(x) = \frac{(3c)^{1/3}}{(x^3+3x)^{1/3}}, \quad x \neq 0.$$