Math 217-001 201810

Practice Assignment # 4 – Answers

1. Solve each equation in two ways: as exact, and by an appropriate substitution.

   (a) $x + y + x\,y' = 0$

   (b) $x - y + x\,y' = 0$

   Answer.

   (a) As exact: the equation is exact, since $\partial(x+y)/\partial y = 1 = \partial x/\partial x$. To find the potential function:

   $$\varphi(x,y) = \int (x+y)\,dx = \frac{x^2}{x} + xy + h(y).$$

   Also,

   $$x = \frac{\partial \varphi}{\partial y} = x + h'(y).$$

   So $h' = 0$ and h is constant. A potential function is then $\varphi(x,y) = x^2/x + xy$, and a solution to the equation satisfies

   $$\frac{x^2}{2} + xy = c$$

   for some constant c. Solving for y, provided that $x \neq 0$, we get

   $$y(x) = \frac{c}{x} - \frac{x}{2}.$$

   By a substitution: if we divide the equation by x we get $1 + y/x + y' = 0$. Letting $u = y/x$, we have $y = xu$, and $y' = u + xu'$. So, in terms of u, the equation looks like $1 + u + u + xu' = 0$, that is

   $$xu' = -1 - 2u.$$

   Now this is separable: $u'/(1+2u) = -1/x$. Taking antiderivatives, $\frac{1}{2}\log(1+2u) = c - \log x$, so $1 + 2u = c/x^2$ (after renaming the constant). That is, $1+2y/x = c/x^2$. Multiplying both sides by x, we get

$$y(x) = \frac{c}{x} - \frac{x}{2}.$$

The choices we made require $x \neq 0$, so the solution will be defined either on $(0,\infty)$ of $(-\infty, 0)$.

(b) As exact: the equation is not exact initially, because $\partial(x-y)/\partial y = -1 \neq 1 = \partial x/\partial x$. To (try to) make this exact, we consider a multiplying factor $\mu(x)$. Now we want $\mu\,(x-y) + \mu\, x\, y' = 0$ to be exact. That is, we need

$$-\mu = \frac{\partial \mu\,(x-y)}{\partial y} = \frac{\partial \mu x}{\partial x} = \mu'\, x + \mu.$$

Then $x\,\mu' = -2\mu$. To solve this separable equation we have $\mu'/\mu = \frac{2}{x}$, or $\log \mu = -2\log x$ (no constant needed, as we need a single μ that gets the job done). Taking exponential on both sides, $\mu(x) = 1/x^2$. So our equation now looks like

$$\frac{1}{x} - \frac{y}{x^2} + \frac{1}{x}\,y' = 0.$$

This is indeed exact: $\partial(1/x - y/x)\partial y = -1/x^2 = \partial(1/x)/\partial x$. We find the potential function:

$$\varphi(x,y) = \int \left(\frac{1}{x} - \frac{y}{x^2}\right)\,dx = \log x + \frac{y}{x} + h(y).$$

Also,

$$\frac{1}{x} = \frac{\partial \varphi}{\partial y} = \frac{1}{x} + h'(y).$$

Then $h' = 0$ and h is constant. Thus $\varphi(x,y) = \log x + y/x$ is a potential function for our equation, and the solutions to our equation satisfy $\log x + y/x = c$ for some constant c. Solving for y,

$$y(x) = cx + x\log x.$$

By using a substitution: if we divide by x, our DE transforms to $1 - y/x + y' = 0$. With the substitution $u = y/x$, we get $xu = y$, and after differentiation $u+xu' = y'$. Substituting in the equation, $1-u+u+xu' = 0$, or

$$u' = -1/x.$$

From there, $u = c - \log x$, and so

$$y(x) = xu = cx - x\log x.$$

The presence of the logarithm requires $x > 0$, so our interval of validity is $(0, \infty)$.

The equation is also linear, so it could have been solve using the integrating factor.

2. Solve the IVP.

   (a) $y' = (x+3y)/(3x+y)$, $y(1) = 1$

   (b) $x^2 + 2y^2 = xy\,y'$, $y(-1) = 1$

   (c) $x\,y' - (1+x)y = xy^2$, $y(1) = 1$

   (d) $y' = \dfrac{1-x-y}{x+y}$, $y(0) = 1$

   Answer.

   (a) Assuming that $x \neq 0$, we divide denominator and numerator by x to get

   $$y' = \frac{1+3y/x}{3+y/x}.$$

   With the substitution $u = y/x$, we have $y' = u + xu'$, and so we have

   $$xu' = \frac{1+3u}{3+u} - u = \frac{1-u^2}{3+u}.$$

   Thus we have the separable equation

   $$\frac{3+u}{1-u^2}\,u' = \frac{1}{x}.$$

   Taking antiderivatives (using partial fractions) we get

   $$\frac{u+1}{(1-u)^2} = cx$$

for some constant c. Solving for u, and recalling that $u = y/x$, we have

$$\frac{y}{x} = \frac{2cx \pm \sqrt{8cx+1} + 1}{2cx},$$

or

$$y(x) = x \pm \frac{\sqrt{8cx+1}}{2c} + \frac{1}{2c}.$$

We have two possible solutions, depending on the choice of the sign. Now using the initial condition on the first solution,

$$2 = y(0) = \pm\frac{1}{2c} + \frac{1}{2c},$$

The negative sign gives us the impossible equality $2 = 0$, so it is not an option. Thus $2 = 1/c$, and $c = 1/2$. The solution is then

$$y(x) = x + \sqrt{4x+1} + 1.$$

(b) If we divide the equation by x^2 we get

$$1 + 2\left(\frac{y}{x}\right)^2 = \frac{y}{x}\, y'.$$

With the substitution $u = y/x$ and $y' = xu' + u$, this becomes $1 + 2u^2 = u(xu' + u)$, or

$$\frac{u}{1+u^2}\, u' = \frac{1}{x}.$$

Taking antiderivatives, $\log(1+u^2) = c + 2\log x$, so $1 + u^2 + cx^2$ (after renaming the constant). Then

$$\frac{y}{x} = \pm\sqrt{cx^2 - 1},$$

or

$$y(x) = \pm\, x\sqrt{cx^2-1}.$$

For the solution with $+$, we have from the initial condition $1 = -\sqrt{c-1}$, impossible. For the solution with $-$, we have from the initial condition $1 = \sqrt{c-1}$, so $c = 2$. The solution is then

$$y(x) = -x\sqrt{2x^2-1}.$$

The interval of validity cannot contain 0 (because we need to divide by x) and has to contain -1. So we can take $(-\infty, 0)$.

(c) This is a Bernoulli equation with $n = 2$. So we use the substitution $v = y^{1-2} = y^{-1}$. By the chain rule, $v' = -y^{-2}\,y' = -v^2\,y'$. So, after this substitution, our equation looks like this:

$$-x\,v^{-2}\,v' - (1+x)v^{-1} = x\,v^{-2}.$$

After multiplying by v^2, we have

$$-x\,v' - (1+x)v = x,$$

while is linear. To find the integrating factor we first need to divide by $-x$, so we have $v' + (1 + 1/x)v = -1$. The integrating factor is then $e^{\int(1+1/x)dx} = e^{x+\log x} = xe^x$. After multiplying by the integrating factor, our equation is

$$(xe^x\,v)' = -xe^x.$$

Taking antiderivatives, $xe^x\,v = (1-x)e^x + c$. So $v(x) = (1-x)/x + ce^{-x}/x$. In terms of $y = v^{-1}$, $y = 1/((1-x)/x + ce^{-x}/x)$. From the initial condition,

$$1 = y(1) = \frac{1}{0 + ce^{-1}} = e/c.$$

So $c = e$. The solution to the IVP is then

$$y(x) = \frac{1}{\frac{1-x}{x} + \frac{e^{1-x}}{x}} = \frac{x}{1 - x + e^{1-x}}.$$

(d) Here we can try the substitution $u = x + y$. Then $u' = 1 + y'$, and the equation becomes $u' - 1 = \frac{1-u}{u}$, which is separable. So $u' = 1 + (1-u)/u = (u + 1 - u)/u = 1/u$. Thus

$$u\,u' = 1.$$

Taking antiderivatives, $u^2/2 = x+c$. Substituting back, $(y+x)^2 = 2x + c$ (after renaming the constant). From the initial condition, $1 = (y(0) + 0)^2 = 2 \times 0 + c$, so $c = 1$. As $(y + x)^2 \geq 0$ always, we need $2x + 1 \geq 0$, i.e. $x \geq -1/2$. From $y(0) = 1$ we also know that $y + x$ will be positive on some interval around 0. So we can solve and we have

$$y(x) = -x + \sqrt{2x+1}, \qquad x \in (-1/2, \infty)$$

(the interval is open at $-1/2$ because the square root is not differentiable at 0).

3. When a vertical beam of light passes through a transparent medium, the rate at which its intensity I decreases is proportional to $I(t)$, where t represents the thickness of the medium (in metres). In clear seawater, the intensity 1 metre below the surface is 25% of the initial intensity I_0 of the incident beam. What is the intensity of the beam 3 metres below the surface?

   Answer. We are told that $I' = kI$. So $I(t)$, where t is the thickness in metres, is

$$I(t) = I_0 \, e^{kt}.$$

   We are also told that $I(1) = I_0/4$. Thus

$$\frac{I_0}{4} = I_0 \, e^k,$$

   and we get $k = -\log 4$. Now we can answer the question: at a depth of 3 metres,

$$I(3) = I_0 e^{-3\log 4} = I_0 \, e^{-\log 4^3} = \frac{I_0}{64}.$$

   So the intensity three metres down is 1/64 of the original intensity, or about 1.56%.

4. The number $N(t)$ of people in a community who are exposed to a particular advertisement is governed by the logistic equation. Initially, $N(0) = 500$, and it is observed that $N(1) = 1000$. Solve for $N(t)$ if it is predicted that the limiting number of people in the community who will see the advertisement is $50,000$.

   Answer. We have $N_0 = N(0) = 500$, and

$$N(t) = \frac{aN_0}{bN_0 + (a - bN_0)e^{-at}}.$$

   We are told that $a/b = 50,000$, and that

$$1000 = N(1) = \frac{aN_0}{bN_0 + (a - bN_0)e^{-a}} = \frac{50,000b \, N_0}{bN_0 + (50,000b - bN_0)e^{-50,000b}}$$

$$= \frac{50,000 \, N_0}{N_0 + (50,000 - N_0)e^{-50,000b}}$$

Then

$$e^{-50,000b} = \frac{1}{50,000 - N_0}\left(\frac{50,000N_0}{1000} - N_0\right) = \frac{49 \times 500}{49,500} = \frac{49}{99},$$

and

$$b = -\frac{1}{50,000}\log\frac{49}{99}, \quad a = 50,000b = -\log\frac{49}{99}.$$

So we have found that

$$N(t) = \frac{-500\log\frac{49}{99}}{-\frac{500}{50,000}\log\frac{49}{99} + (-\log\frac{49}{99} + \frac{500}{50,000}\log\frac{49}{99})e^{t\log\frac{49}{99}}}$$

$$= \frac{50,000}{1 + 99\left(\frac{49}{99}\right)^t}$$

5. A model for the population $P(t)$ in a suburb of a large city is given by the initial-value problem

$$P' = P(10^{-1} - 10^{-7}P), \quad P(0) = 5000,$$

where t is measured in months. What is the limiting value of the population? At what time will the population be equal to one-half of this limiting value?

Answer. We have $a = 10^{-1}$, $b = 10^{-7}$. The limiting value is

$$\frac{a}{b} = \frac{10^{-1}}{10^{-7}} = 10^6.$$

To find when the population is half of the limiting value, that is $a/2b = 500,000$, we solve

$$500,000 = \frac{10^{-1} \times 5,000}{10^{-7} \times 5,000 + (10^{-1} - 10^{-7} \times 5,000)e^{-10^{-1}t}}$$

$$= \frac{500}{\frac{5}{10^4} + \left(\frac{1}{10} - \frac{5}{10^4}\right)e^{-t/10}}$$

$$= \frac{5,000,000}{5 + 995e^{-t/10}}.$$

So

$$e^{-t/10} = \frac{10-5}{995} = \frac{5}{995} = \frac{1}{199},$$

and

$$t = -10 \log \frac{1}{199} = 10 \log 199 \simeq 52.9 \quad \text{months.}$$