Math 217-001 201810
Practice Assignment # 5 – Answers

1. (a) Show that $c_1 + c_2x^2$ is a two-parameter family of solutions of the equation $xy'' - y' = 0$, $x \in (-\infty, \infty)$.

   (b) Show that the initial value problem

$$\begin{cases} xy'' - y' = 0, \\ y(0) = 0, \\ y'(0) = 1 \end{cases}$$

   has no solution.

   (c) Does the fact that the above IVP has no solution contradict the Existence and Uniqueness Theorem (4.1.1 in the textbook)?

Answer.

(a) We compute

$$x(c_1 + c_2x^2)'' - (c_1 + c_2x^2)' = 2c_2x - 2c_2x = 0.$$

(b) A solution is of the form $y(x) = c_1 + c_2x^2$. Looking at the second initial value,

$$1 = y'(0) = 2c_2 \times 0 = 0,$$

which is impossible. So the IVP has no solution.

(c) There is no contradiction. The theorem considers the equation in its form $y'' - \frac{1}{x}y' = 0$. But then the coefficient of y' is not continuous at zero, which a requirement of the theorem.

2. (a) Show that $c_1 \cos x + c_2 \sin x$ is a two-parameter family of solutions of the equation $y'' + y = 0$.

   (b) Consider the BVP

$$\begin{cases} y'' + y = 0, \\ y(0) = 0, \\ y(\pi) = 0 \end{cases}$$

   Show that the system has infinitely many solutions.

(c) Does the fact that the above BVP has no solution contradict the Existence and Uniqueness Theorem (4.1.1 in the textbook)?

Answer.

(a) We compute

$$(c_1 \cos x + c_2 \sin x)'' + c_1 \cos x + c_2 \sin x = -c_1 \cos x - c_2 \sin x + c_1 \cos x + c_2 \cos x = 0.$$

(b) Looking at the boundary values, $0 = y(0) = c_1$, so we know that $c_1 = 0$. Looking at the second boundary value, $0 = y(\pi) = c_2 \sin \pi$. As $\sin \pi = 0$, any c_2 works. So any function

$$y(x) = c_2 \sin x,$$

for any number c_2, is a solution to the BVP.

(c) There is no contradiction because the theorem considers only Initial Value Problems, and not Boundary Value Problems like this one.

3. (a) Show that $y(x) = e^{3x}$ is a particular solution of $y'' - 2y' + 5y = 8e^{3x}$.

(b) Show that $y(x) = 4x^3 + 2x$ is a particular solution of $y'' - 2y' + 5y = 20x^3 - 24x^2 + 34x - 4$.

(c) Find a particular solution of $y'' - 2y' + 5y = e^{3x} - 20x^3 + 24x^2 - 34x + 4$.

Answer.

(a) We compute

$$(e^{3x})'' - 2(e^{3x})' + 5e^{3x} = 9e^{3x} - 6e^{3x} + 5e^{3x} = 8e^{3x}.$$

(b) We compute

$$\begin{aligned}(4x^3 + 2x)'' - 2(4x^3 + 2x)' + 5(4x^3 + 2x) &= 24x - 24x^2 - 4 + 20x^3 + 10x \\ &= 20x^3 - 24x^2 + 34x - 4.\end{aligned}$$

(c) Since $(e^{3x})'' - 2(e^{3x})' + 5e^{3x} = 8e^{3x}$, we have that

$$(\frac{1}{8}e^{3x})'' - 2(\frac{1}{8}e^{3x})' + \frac{5}{8}e^{3x} = e^{3x}$$

And since $(4x^3+2x)''-2(4x^3+2x)'+5(4x^3+2x) = 20x^3-24x^2+34x-4$,

$$(-4x^3-2x)''-2(-4x^3-2x)'+5(-4x^3-2x) = -20x^3+24x^2-34x+4.$$

If we add this two particular solutions, we get

$$y(x) = \frac{1}{8}e^{3x} - 4x^3 - 2x,$$

which is a solution of $y'' - 2y' + 5y = e^{3x} - 20x^3 + 24x^2 - 34x + 4$.

4. Verify that the given functions form a fundamental set of solutions of the given equation on the indicated interval. Solve the IVP.

   (a) $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$; $\cos 2x$, $\sin 2x$, $(-\infty, \infty)$.

   (b) $4y'' - 4y' + y = 0$, $y(0) = 4$, $y'(0) = 1$; $e^{x/2}$, $xe^{x/2}$, $(-\infty, \infty)$.

   Answer.

   (a) We have

   $$(\cos 2x)'' + 4\cos 2x = -4\cos 2x + 4\cos 2x = 0,$$

   and similarly for $\sin 2x$. So both are solutions to the equation. They are linearly independent, since the Wronskian is

   $$W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2 \neq 0.$$

   The general solution is then $y(x) = c_1 \cos 2x + c_2 \sin 2x$ for constants c_1, c_2. Looking at the initial conditions,

   $$1 = y(0) = c_1, \quad 1 = y'(0) = 2c_2.$$

   So the solution to the IVP is $y(x) = \cos 2x + \frac{1}{2}\sin 2x$.

   (b) We compute

   $$4(e^{x/2})'' - 4(e^{x/2})' + e^{x/2} = e^{x/2} - 2e^{x/2} + e^{x/2} = 0,$$

   $$4(xe^{x/2})''-4(xe^{x/2})'+xe^{x/2} = 4e^{x/2}+xe^{x/2}-4e^{x/2}-2xe^{x/2}+xe^{x/2} = 0.$$

So both are solutions. Linear independence:

$$W(x) = \begin{bmatrix} e^{x/2} & xe^{x/2} \\ \frac{1}{2}\, e^{x/2} & e^{x/2} + \frac{x}{2}\, e^{x/2} \end{bmatrix} = e^x(1 + \frac{x}{2} - \frac{x}{2}) = e^x \neq 0.$$

So the two solutions are linearly independent and the general solution is thus $y(x) = c_1 e^{x/2} + c_2 x e^{x/2}$. Looking at the initial values,

$$4 = y(0) = c_1, \quad 1 = y'(0) = \frac{c_1}{2} + c_2 = 2 + c_2,$$

so $c_1 = 4$, $c_2 = -1$. The solution to the IVP is then $y(x) = 4e^{x/2} - xe^{x/2} = (4 - x)e^{x/2}$.

5. The function $y_1 = e^{-3x}$ is a solution of the associated homogeneous equation of the equation $y'' - 9y = 2$. Using reduction of order,

   (a) Find a second solution y_2 to the homogeneous equation;

   (b) Check that y_1 and y_2 form a fundamental set of solutions of the homogeneous equation.

   (c) Find a particular solution y_p to the equation $y'' - 9y = 2$.

   (d) Express the general solution of $y'' - 9y = 2$.

   Answer.

   (a) We postulate $y_2 = uy_1 = ue^{-3x}$. If y_2 is a solution to the homogeneous equation, then

   $$\begin{aligned} 0 = y_2'' - 9y_2 &= u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x} - 9ue^{-3x} \\ &= u''e^{-3x} - 6u'e^{-3x} \\ &= e^{-3x}\,(u'' - 6u'). \end{aligned}$$

   so u satisfies the equation $u'' - 6u' = 0$. That is, $u''/u' = 6$, separable. Taking antiderivatives, $\log u' = 6x + c$, and then $u' = d\,e^{6x}$. Thus $u(x) = \frac{d}{6}\, e^{6x}$, and taking $d = 6$ we can use $u(x) = e^{6x}$. This gives $y_2(x) = u(x)y_1(x) = e^{3x}$.

   (b) We check that e^{3x} and e^{-3x} are linearly independent:

   $$W(x) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0.$$

   So e^{3x} and e^{-3x} form a fundamental set of solutions of $y'' - 9y = 0$.

(c) Now we use reduction of order on the equation $y'' - 9y = 2$, starting again from y_1. The computations are the same as before, only that our equation is equated to 2 instead of 0. So we get

$$2 = e^{-3x}\,(u'' - 6u'),$$

or $u'' - 6u' = 2e^{3x}$. This is linear, with integrating factor $\mu = e^{\int(-6)dx} = e^{-6x}$. After multiplication by the integrating factor, the equation becomes

$$(e^{-6x}u')' = 2e^{-3x}.$$

Looking at the antiderivatives, $e^{-6x}u' = -\frac{2}{3}\,e^{-3x}$, which gives $u' = -\frac{2}{3}\,e^{3x}$. Taking antiderivatives once more, we get $u(x) = -\frac{2}{9}\,e^{3x}$. This gives us the particular solution

$$y_p(x) = u(x)y_1(x) = -\frac{2}{9}\,e^{3x}\,e^{-3x} = -\frac{2}{9}.$$

(d) The general solution of $y'' - 9y = 2$ is then

$$y(x) = c_1 e^{3x} + c_2 e^{-3x} - \frac{2}{9},$$

for constants c_1 and c_2.

6. Decide if the functions are linearly independent.

(a) $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = x^3$.
Answer. Using the Wronskian:

$$\begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3 \neq 0.$$

Here is another way: if $c_1x + c_2x^2 + c_3x^3 = 0$, using three different values of x, say $x = 1$, $x = 2$, $x = 3$, we get a linear system of 3 equations on the three unknowns c_1, c_2, c_3,

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 + 4c_2 + 8c_3 &= 0 \\ 3c_1 + 9c_2 + 27c_3 &= 0 \end{aligned}$$

Since the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 12 \neq 0$, the system has a unique solution, which has to be $c_1 = c_2 = c_3 = 0$. So the three functions are linearly independent.

(b) $f_1(x) = \sin x,\ f_2(x) = \cos x$.

Answer. If $c_1 \sin x + c_2 \cos x = 0$, we can take $x = \pi/2$ and we get $c_1 = 0$. Similarly, with $x = 0$ we get $c_2 = 0$. So the two functions are linearly independent.

We could have also calculated the Wronskian,

$$\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0.$$

(c) $f_1(x) = \sin^2 x,\ f_2(x) = \cos^2 x$.

Answer. We can use the same idea as before. If $c_1 \sin^2 x + c_2 \cos^2 x = 0$, we take alternatively $x = \pi/2$, $x = 0$ to get that $c_1 = c_2 = 0$. So the two functions are linearly independent. The Wronskian is not particularly enlightening in this example, but it can also easily seen that it is not zero.

(d) $f_1(x) = \sinh x,\ f_2(x) = \cosh x$.

Answer. We can easily calculate the Wronskian:

$$\begin{vmatrix} \sinh x & \cosh x \\ \cosh x & \sinh x \end{vmatrix} = \sinh^2 x - \cosh^2 x = \frac{(e^x - e^{-x})^2 - (e^x + e^{-x})^2}{4}$$
$$= \frac{-2-2}{4} = -1.$$

Thus f_1 and f_2 are linearly independent.

7. Solve the following initial value problems:

(a) $y'' + 2y' + y = 0$, $y(1) = 2$, $y'(1) = -2$;

Answer. This is a linear, homogeneous, constant coefficient equation. The characteristic equation is $k^2 + 2k + 1 = 0$, which is $(k+1)^2 = 0$ and has the only solution $k = -1$. So $y_1(x) = e^{-x}$ is a solution to the equation. For the other solution to complete a fundamental set of solutions we take $y_2(x) = xe^{-x}$.

How do we know how to choose y_2*?By reduction of order. We postulate a solution of the form* $y_2(x) = u(x)y_1(x)$ *and we find* u*. If* y_2 *satisfies the equation, we should have*

$$\begin{aligned} 0 &= y_2'' + 2y' + y = u''y_1 + 2u'y_1' + uy_2'' + 2u'y_1 + 2uy_1' + uy_1 \\ &= u''y_1 + 2u'(y_1' + y_1) = u''e^{-x} + 2u'(-e^{-x} + e^{-x}) = u''e^{-x}; \end{aligned}$$

So $u'' = 0$*, that is* $u = x$*. Then* $y_2(x) = u(x)y_1(x) = xe^{-x}$*.*

The general solution is thus $c_1e^{-x} + c_2x\,e^{-x}$. The condition $y(1) = 2$ translates into $c_1e^{-1} + c_2e^{-1} = 2$; and $y'(1) = -2$ is $-2 = -c_1e^{-1} + c_2(1-1)e^{-1} = -c_1e^{-1}$. So $c_1 = 2e$ and (from the first equation) $c_2 = 2e - c_1 = 2e - 2e = 0$. The solution to the IVP is then $y(x) = 2e\,e^{-x} = 2e^{1-x}$.

(b) $y'' + 2y' = 0$, $y(1) = 2$, $y'(1) = -2$;

Answer. This is again a linear, homogeneous, constant coefficient equation. The characteristic equation is $k^2 + 2k = 0$, which has the two real solutions $k = 0$, $k = -2$. So the functions $y_1(x) = e^0 = 1$ and $y_2(x) = e^{-2x}$ are a fundamental set of solutions. The general solution is then $y(x) = c_1 + c_2e^{-2x}$. The initial conditions translate into $c_1 + c_2e^{-2} = 2$, and $-2c_2e^{-2} = -2$. So $c_2 = e^2$ and $c_1 = 2 - c_2e^{-2} = 2 - 1 = 1$. The general solution is then $y(x) = 1 + e^2\,e^{-2x} = 1 + e^{2-2x}$.

(c) $y'' + 2y' + 2y = 0$, $y(1) = 2$, $y'(1) = -2$.

Answer. This is again a linear, homogeneous, constant coefficient equation. The characteristic equation is $k^2 + 2k + 2 = 0$. This equation has no real solutions: its two complex solutions are $-1 + i$ and $-1 - i$. Then a fundamental set of solutions will be given by $y_1(x) = e^{-x}\cos(x)$, $y_2 = e^{-x}\sin(x)$. The general solution is then $y(x) = c_1e^{-x}\cos(x) + c_2e^{-x}\sin(x)$. The two initial conditions give the equations $c_1e^{-1}\cos(1) + c_2e^{-1}\sin(1) = 2$, $(c_2 - c_1)e^{-1}\cos(1) - (c_1 + c_2)e^{-1}\sin(1) = -2$. If we multiply each equation by e and then we add them, we get $c_2\,\cos(1) - c_1\,\sin(1) = 0$. So $c_2 = c_1\,\sin(1)/\cos(1)$. Putting this into the first equation we get $c_1\,\cos(1) + c_1\,\sin^2(1)/\cos(1) = 2e$, which gives $c_1 = 2e\cos(1)$. And then $c_2 = c_1\,\sin(1)/\cos(1) = 2e\sin(1)$. The solution to the IVP is then

$$\begin{aligned} y(x) &= 2e\cos(1)e^{-x}\cos(x) + 2e\sin(1)e^{-x}\sin(x) \\ &= 2e^{1-x}(\cos(1)\cos(x) + \sin(1)\sin(x)) \\ &= 2e^{1-x}\,\cos(1-x). \end{aligned}$$

(d) $x^2y'' + 3xy' + y = 0$, $y(1) = 2$, $y'(1) = -2$;

Answer. This is an Euler equation. It's characteristic equation is $k^2 + 2k + 1 = 0$, it has one solution, $k = -1$ (because the equation is $(k+1)^2 = 0$). So one solution for the equation is $y_1(x) = x^{-1} = 1/x$. We can get another linearly independent solution by $y_2(x) = \ln x\, y_1(x) = (\ln x)/x$.

How do we find *y_2? By Reduction of Order. Assume $y_2 = uy_1$ is a solution. Then*

$$\begin{aligned} 0 &= x^2y_2'' + 3xy_2' + y_2 = x^2u''y_2 + 2x^2u'y_1' + x^2uy_1'' + 3xu'y_1 + 3xuy_1' + uy_1 \\ &= x^2u''y_1 + 2x^2u'y_1' + 3xu'y_1 = x^2u''y_1 + u'(2x^2y_1' + 3xy_1) = xu'' + u'(-2+3) \\ &= xu'' + u'; \end{aligned}$$

then $u''/u' = -1/x$, so $\ln u' = -\ln x = \ln(1/x)$, $u' = -1/x$, $u = \ln x$.

So the general solution can be written as $y(x) = c_1/x + c_2\,(\ln x)/x = (c_1 + c_2 \ln x)/x$. Note that

$$y'(x) = -c_1/x^2 + c_2(1 - (\ln x)^2)/x^2 = (c_2 - c_1 - 2(\ln x)^2)/x^2.$$

For the IVP we have $2 = y(1) = c_1/1 + c_2\ln(1)/1 = c_1$, and $-2 = y'(1) = (c_2 - 2\ln(1)^2)/1^2 = c_2 - 2$. So $c_2 = 0$ and the solution to the IVP is $y(x) = 2/x$.

(e) $x^2y'' + 3xy' = 0$, $y(1) = 2$, $y'(1) = -2$;

Answer. Again an Euler equation, with characteristic equation $k^2 + 2k = 0$. So $k = 0$, $k = -2$ are the two solutions. So $y_1(x) = x^0 = 1$, $y_2(x) = x^{-2}$ is a fundamental system of solutions, and the general solution is then $y(x) = c_1 + c_2/x^2$. The conditions of the IVP give $2 = y(1) = c_1 + c_2$ and $-2 = y'(1) = -2c_2/1^3 = -2c_2$. Then $c_2 = 1$ and $c_1 = 2 - c_2 = 1$. The solution of the IVP is then $y(x) = 1 + 1/x^2$.

(f) $x^2y'' + 3xy' + 2y = 0$, $y(1) = 2$, $y'(1) = -2$.

Answer. In this Euler equation the characteristic equation is $k^2 + 2k + 2 = 0$, with complex solutions $-1 + i$, $-1 - i$. So we get a fundamental system of solutions $y_1(x) = \cos(\ln x)/x$, $y_2(x) = \sin(\ln x)/x$. We write the general solution as $y(x) = (c_1\cos(\ln x) + c_2\sin(\ln x))/x$. We have

$$y'(x) = \frac{(c_2 - c_1)\cos(\ln x) - (c_1 + c_2)\sin(\ln x)}{x^2}.$$

The IVP conditions are $2 = y(1) = (c_1 \cos(0) + c_2 \sin(0))/1 = c_1$ and $-2 = y'(1) = (c_2 - c_1)\cos(0) - (c_1 + c_2)\sin(0) = c_2 - c_1$. Then $c_2 = -2 + c_1 = 0$ and the solution to the IVP is

$$y(x) = \frac{2\cos(\ln x)}{x}.$$

8. Show that $y_1(x) = 3e^{2x} - 1$ and $y_2(x) = e^{-x} + 2$ are solutions of $y\,y'' + 2y' - (y')^2 = 0$ but that neither $2y_1$ nor $y_1 + y_2$ is a solution. On the other hand, we proved in class that scalar multiplies and sums of solutions of linear DE are again solutions. Is there a contradiction ?

   Answer. To verify they are solutions:

$$\begin{aligned} y_1 y_1'' + 2y_1' - (y_1')^2 &= (3e^{2x} - 1)12e^{2x} + 2 \times 6e^{2x} - (6e^{2x})^2 \\ &= e^{2x}\,(36e^{2x} - 12 + 12 - 36e^{2x}) = 0. \end{aligned}$$

   Similarly,

$$\begin{aligned} y_2 y_2'' + 2y_2' - (y_2')^2 &= (e^{-x} + 2)e^{-x} + 2 \times (-e^{-x}) - (-e^{-x})^2 \\ &= e^{-x}\,(e^{-x} + 2 - 2 - e^{-x}) = 0. \end{aligned}$$

   On the other hand,

$$\begin{aligned} 2y_1(2y_1)'' + 2(2y_1)' - (2y_1')^2 &= (6e^{2x} - 2)24e^{2x} + 2 \times 12e^{2x} - (12e^{2x})^2 \\ &= e^{2x}\,(144e^{2x} - 48 + 24 - 144e^{2x}) \\ &= -24e^{2x}, \end{aligned}$$

   and if $y_3 = y_1 + y_2$, then

$$y_3 y_3'' + 2y_3' - (y_3')^2 = 36e^{4x} - 12e^{2x} + 21e^{x} + 5e^{-x} \neq 0.$$

   There is no contradiction because the equation is not linear.