University of Regina
Department of Mathematics and Statistics
Math 217-001 201810
Midterm 1 – Answers

Date: February 16th, 2018
Duration: 50 minutes

1. Solve the IVP

$$\begin{cases} xy' + (2x+1)y = e^{-3x}, \\ \\ y(1) = 0 \end{cases}$$

Answer. This is linear equation, when we write it as

$$y'' + \left(2 + \frac{1}{x}\right) y = \frac{1}{x}\, e^{-3x}.$$

This requires $x \neq 0$, so we take $x > 0$ to include the initial point. The integrating factor is

$$\mu(x) = e^{\int (2+1/x)dx} = e^{2x+\log x} = xe^{2x}.$$

Thus, after multiplying by xe^{2x}, the equation becomes

$$(xe^{2x}\, y)' = e^{-x}.$$

So $xe^{2x}y = c - e^{-x}$ for some constant c. From $y(1) = 0$ we get $0 = c - e^{-1}$, so $c = e^{-1}$. Then $xe^{2x}y = e^{-1} - e^{-x}$, and solving for y we obtain

$$y(x) = \frac{e^{-2x-1} - e^{-3x}}{x}, \quad x > 0.$$

2. Solve the IVP

$$\begin{cases} y' = \dfrac{1-x-y}{x+y}, \\ \\ y(0) = 1 \end{cases}$$

Answer. If we substitute $u = x + y$, then $u' = 1 + y'$, and the equation becomes

$$u' - 1 = \frac{1-u}{u},$$

or $u' = 1/u$. Solving the separable equation $uu' = 1$, we can write this as $u^2/2 = x + c$. From the initial condition, $1/2 = c$. Substituting back and multiplying by 2, $(x+y)^2 = 2x + 1$. Then $2x + 1 \geq 0$, so $x \geq -1/2$. As y has to be positive to have $y(0) = 1$, we solve as

$$y(x) = -x + \sqrt{2x+1}, \quad x > -1/2.$$

3. Solve the IVP

$$\begin{cases} (e^x + y) + (2 + x + e^y)y' = 0, \\ \\ y(0) = 1 \end{cases}$$

Answer. We test for exactness: with $P = e^x + y$, $Q = 2 + x + e^y$ we have

$$\frac{\partial P}{\partial y} = 1, \quad \frac{\partial Q}{\partial x} = 1.$$

So the equation is exact. We look for the potential function φ. From $P = \frac{\partial \varphi}{\partial x}$ we get

$$\varphi(x, y) = e^x + xy + h(y).$$

Then

$$2 + x + e^y = \frac{\partial \varphi}{\partial y} = x + h'(y).$$

So $h'(y) = 2 + e^y$, i.e. $h(y) = 2y + e^y + c$ for some constant c. A solution to the equation then satisfies $\varphi(x, y) = 0$, that is

$$e^x + xy + 2y + e^y + c = 0.$$

From $y(0) = 1$, $0 = 1 + 2 + e + c$. So $c = -e - 3$, and y is expressed implicitly by

$$e^x + xy + 2y + e^y = e + 3, \quad -\infty < x < \infty.$$

4. Initially 100 milligrams of a radioactive substance was present. After 6 hours the mass had decreased by 3%. If the rate of decay is proportional to the amount of the substance present at time t,

   (a) find the amount remaining after 24 hours;

   (b) Determine the half-life.

Answer. If $S(t)$ is the amount of material at time t, we have $S'(t) = kS(t)$ for a certain constant k. We are also told that $S(0) = 100$. So

$$S(t) = 100e^{kt}.$$

If we measure t in hours, we are told that $S(6) = 97$. So

$$97 = 100\, e^{6k},$$

which gives

$$k = \frac{1}{6} \log \frac{97}{100}.$$

The amount after 24 hours is then

$$S(24) = 100\, e^{\frac{24}{6} \log \frac{97}{100}} = 100\, e^{4 \log \frac{97}{100}} = 100 \left(\frac{97}{100} \right)^4 \simeq 88.529.$$

For the half life, we want to solve

$$100\, e^{\frac{t}{6}\, \log \frac{97}{100}} = 50.$$

That is,

$$\frac{t}{6}\, \log \frac{97}{100} = \log \frac{1}{2} = -\log 2,$$

which gives

$$t = -\frac{6 \log 2}{\log \frac{97}{100}} \simeq 136.54.$$