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A contractive version of a Schur–Horn theorem in II_1 factors

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Abstract

We prove a contractive version of the Schur–Horn theorem for submajorization in II_1 factors that complements some previous results on the Schur–Horn theorem within this context. We obtain a reformulation of a conjecture of Arveson and Kadison regarding a strong version of the Schur–Horn theorem in II_1 factors in terms of submajorization and contractive orbits of positive operators.

Keywords: Majorization; Submajorization; Contractive orbits; Schur–Horn theorem

1. Introduction

Vector and matrix majorization theory play an important role in matrix analysis, mostly as ~~a~~ tool^ in the study of general (convex) inequalities, unitarily invariant norm inequalities, geometry, and problems related with the description of the diagonals of matrix representations of a linear operator [1,2,4,12]. Some historical aspects of the theory of majorization are mentioned in [3,5]. The Schur–Horn theorem, coined in the papers [8,15], is probably the most remarkable among the many characterizations known for these notions (see the precise statement of the the-

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orem after Proposition 2.3). It is thus natural to search for analogues of this result in contexts where majorization theory has been extended [5–7,10,13]. Among these analogues let us mention the work of Neumann [13] for selfadjoint operators in $\mathcal{B}(\mathcal{H})$, the refinements of Kadison [11] in the case of projectors in $\mathcal{B}(\mathcal{H})$, and the recent work [5].

The fact that II_1 factors share many structural properties with the algebra of linear operators acting on $\mathbb{C}^n$ makes them a natural context in which to extend majorization. In [5], Arveson and Kadison posed a (strong) version of the Schur–Horn theorem for II_1 factors as a problem and proved related results. As a first step toward settling the Arveson–Kadison problem, the authors have proven in [3] a weaker version, related with the point of view developed in [13]. In this note we obtain a weak contractive version of submajorization within II_1 factors in the spirit of [3] (Theorem 3.4). We also obtain an equivalent reformulation of the Arveson–Kadison problem (Theorem 4.1) using a characterization of spectral dominance and submajorization (Proposition 3.1).

2. Preliminaries

Throughout the paper $\mathcal{M}$ denotes a II_1 factor with normalized faithful normal trace τ. We denote by $\mathcal{M}^{\mathrm{sa}}$, $\mathcal{M}^+$, $\mathcal{U}_{\mathcal{M}}$, the sets of selfadjoint, positive, and unitary elements of $\mathcal{M}$. Given $a \in \mathcal{M}^{\mathrm{sa}}$ we denote its spectral measure by p^a. The characteristic function of the set Δ is denoted by 1_Δ. We denote integration with respect to Lebesgue measure by dt.

Besides the usual operator norm in $\mathcal{M}$, we consider the Schatten norm induced by the trace, $\|x\|_1 = \tau(|x|)$. As we will be always dealing with bounded sets in a II_1 factor, we can profit from the fact that the topology induced by the Schatten norm agrees with the σ-strong operator topology. Because of this we will express our results in terms of σ-strong closures although our computations are based on estimates for the Schatten norm. For $X \subset \mathcal{M}$, we shall denote by $\overline{X}$ and $\overline{X}^{\sigma\text{-sot}}$ the respective closures in the norm topology and in the σ-strong operator topology. For any set K, $\operatorname{co} K$ denotes its convex hull.

2.1. Spectral scale and spectral preorders

The *spectral scale* [14] of $a \in \mathcal{M}^{\mathrm{sa}}$ is defined by

$$\lambda_a(t) = \min\bigl\{s \in \mathbb{R}\colon \tau\bigl(p^a(s,\infty)\bigr) \leqslant t\bigr\}, \quad t \in [0,1).$$

The function $\lambda_a : [0,1) \to \mathbb{R}$ is non-increasing and right-continuous. The map $a \mapsto \lambda_a$ is continuous both with respect to $\|\cdot\|$ and $\|\cdot\|_1$, since [14]

$$\|\lambda_a - \lambda_b\|_\infty \leqslant \|a-b\|, \quad \|\lambda_a - \lambda_b\|_1 \leqslant \|a-b\|_1, \quad a, b \in \mathcal{M}^{\mathrm{sa}}, \tag{1}$$

where the norms on the left are those of $L^\infty([0,1], dt)$ and $L^1([0,1], dt)$, respectively. A useful property of the spectral scale is that we can use it to recover the trace, in the following sense:

$$\tau(a) = \int_0^1 \lambda_a(t)\, dt. \tag{2}$$

The unitary orbit of $a \in \mathcal{M}^{\mathrm{sa}}$ is the set $\mathcal{U}_{\mathcal{M}}(a) = \{u^*au\colon u \in \mathcal{U}_{\mathcal{M}}\}$. It is straightforward from the definition of the spectral scale that if $b \in \mathcal{U}_{\mathcal{M}}(a)$, then $\lambda_a = \lambda_b$. By the continuity (1), $\lambda_b = \lambda_a$ for any b in the $\|\cdot\|_1$-closure or the $\|\cdot\|$-closure of the unitary orbit of $a \in \mathcal{M}^{\mathrm{sa}}$. A converse of this fact was proven by Kamei. We summarize this information for future reference:

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Theorem 2.1. *(~~See~~ [9]~~.~~) If $a \in \mathcal{M}^{sa}$, then*

$$\overline{\mathcal{U}_{\mathcal{M}}(a)} = \overline{\mathcal{U}_{\mathcal{M}}(a)}^{\sigma\text{-sot}} = \{b \in \mathcal{M}^{sa}: \lambda_a = \lambda_b\}.$$

Let $a, b \in \mathcal{M}^{sa}$. We say that a is *spectrally dominated* by b, written $a \precsim b$, if any of the following (equivalent) statements holds:

(i) $\lambda_a(t) \leqslant \lambda_b(t)$, for all $t \in [0,1]$.
(ii) $\tau(p^a(t,\infty)) \leqslant \tau(p^b(t,\infty))$, for all t.

We say that a is *submajorized* by b, written $a \prec_w b$, if

$$\int_0^s \lambda_a(t)\,dt \leqslant \int_0^s \lambda_b(t)\,dt, \quad \text{for every } s \in [0,1).$$

If in addition $\tau(a) = \tau(b)$, then we say that a is *majorized* by b, written $a \prec b$.

Remark 2.2. Let $a, b \in \mathcal{M}^{sa}$. It is known [14] that

(i) if $a \leqslant b$, then $a \precsim b$. Thus, using this and (2),

$$a \leqslant b \quad \Rightarrow \quad a \precsim b \quad \Rightarrow \quad a \prec_w b \quad \Rightarrow \quad \tau(a) \leqslant \tau(b);$$

(ii) if $v \in \mathcal{M}$ is a contraction ($\|v\| \leqslant 1$), then $v^*av \precsim a$.

If $\mathcal{N} \subset \mathcal{M}$ is a von Neumann subalgebra and $b \in \mathcal{M}^{sa}$, we denote by $\Omega_{\mathcal{N}}(b)$ and $\Theta_{\mathcal{N}}(b)$ the sets of elements in $\mathcal{N}^{sa}$ that are, respectively, majorized and submajorized by b, i.e.,

$$\Omega_{\mathcal{N}}(b) = \{a \in \mathcal{N}^{sa}: a \prec b\}, \qquad \Theta_{\mathcal{N}}(b) = \{a \in \mathcal{N}^{sa}: a \prec_w b\}.$$

The following result was proven in [3].

Proposition 2.3. *Let $b \in \mathcal{B}^{sa}$, where $\mathcal{B} \subset \mathcal{M}$ is a diffuse abelian von Neumann subalgebra. Then there exists a spectral resolution $\{e(t)\}_{t\in[0,1]} \subset \mathcal{B}$ with $\tau(e(t)) = t$, for every $t \in [0,1]$, and such that*

$$b = \int_0^1 \lambda_b(t)\,de(t).$$

The classical Schur–Horn theorem states that if $\mathcal{N}$ is a type I_n factor, $\mathcal{D} \subset \mathcal{N}$ is a masa, $E_{\mathcal{D}}$ is the canonical projection onto $\mathcal{D}$, and $b \in \mathcal{N}^{sa}$, then

$$E_{\mathcal{D}}(\mathcal{U}_{\mathcal{N}}(b)) = \Omega_{\mathcal{D}}(b).$$

In [3], the authors proved the following related result for II_1 factors.

Theorem 2.4. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra and let $b \in \mathcal{M}^{sa}$. Then*

$$\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} = \Omega_{\mathcal{A}}(b). \tag{3}$$

3. A contractive version of the Schur–Horn theorem

Given $x \in \mathcal{M}$ we shall consider its contractive orbit $\mathcal{C}_{\mathcal{M}}(x)$, namely

$$\mathcal{C}_{\mathcal{M}}(x) := \left\{ v^* x v \colon v \in \mathcal{M},\ \|v\| \leqslant 1 \right\}.$$

Using the results quoted in Section 2, we prove the following characterization of submajorization and spectral dominance.

Proposition 3.1. *Let $a, b \in \mathcal{M}^+$. Then*

(i) *$a \prec_w b$ if and only if there exists $c \in \mathcal{M}^+$ such that $a \prec c \leqslant b$. Moreover, if $\mathcal{B} \subset \mathcal{M}$ is a diffuse abelian von Neumann subalgebra such that $b \in \mathcal{B}^+$, we can choose $c \in \mathcal{B}^+$.*
(ii) *$a \precsim b$ if and only if $a \in \overline{\mathcal{C}_{\mathcal{M}}(b)}$.*

Proof. (i) Assume first that $a \prec_w b$ and, without loss of generality, assume that $b \in \mathcal{B}$ for a diffuse abelian subalgebra $\mathcal{B} \subseteq \mathcal{M}$. Let $\{e(t)\}_{t \in [0,1]} \subseteq \mathcal{B}$ be a spectral resolution as in Proposition 2.3. Since the function $g(s) := \int_0^s \lambda_b(t)\, dt$ is continuous and $a \prec_w b$, there exists $s_0 \in [0,1]$ such that $\tau(a) = g(s_0)$. Thus, if we let $c = \int_0^{s_0} \lambda_b(t)\, de(t)$, it is straightforward to verify that $\lambda_c(t) = 1_{[0,s_0]} \lambda_b(t)$ for $t \in [0,1)$. From this it follows that $a \prec c$. It is also clear that $c \in \mathcal{B}$ and that $c \leqslant b$. Conversely, if there exists $c \in \mathcal{M}^+$ such that $a \prec c \leqslant b$, then $a \prec_w c$ and $c \prec_w b$, and so by transitivity we get $a \prec_w b$.

(ii) Let $a, b \in \mathcal{M}^+$ with $a \precsim b$. Let $\mathcal{B}$ be a diffuse abelian subalgebra with $b \in \mathcal{B}$, and let $\{e(t)\}_{t \in [0,1]} \subseteq \mathcal{B}$ be as before. By hypothesis $0 \leqslant \lambda_a \leqslant \lambda_b$, so in particular $\{\lambda_b = 0\} \subseteq \{\lambda_a = 0\}$. Thus the function $f = 1_{\{\lambda_b \neq 0\}} \cdot \lambda_a / \lambda_b$ is well defined, $0 \leqslant f \leqslant 1$, and $f \cdot \lambda_b = \lambda_a$. Therefore, $v = \int_0^1 f(t)^{1/2}\, de(t) \in \mathcal{B}$ is a contraction such that

$$v^* b v = \int_0^1 \lambda_a(t)\, de(t) \quad \text{and thus} \quad \lambda_{v^* b v} = \lambda_a.$$

By Theorem 2.1 it follows that $a \in \overline{\mathcal{U}_{\mathcal{M}}(v^* b v)} \subset \overline{\mathcal{C}_{\mathcal{M}}(b)}$. To see the converse, let $a \in \overline{\mathcal{C}_{\mathcal{M}}(b)}$. Then $a \precsim b$ since, by (ii) in Remark 2.2, $v^* b v \precsim b$ for any contraction $v \in \mathcal{M}$, and by (1) the spectral scale is uniformly continuous with respect to the operator norm. □

In [6, Theorem 3.1], Hiai shows that $\{a \in \mathcal{M} \colon a \precsim b\} = \overline{\mathcal{C}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}$. So, from Proposition 3.1, we obtain

Corollary 3.2. *If $b \in \mathcal{M}$, then $\overline{\mathcal{C}_{\mathcal{M}}(b)}^{\sigma\text{-sot}} = \overline{\mathcal{C}_{\mathcal{M}}(b)}$.*

Lemma 3.3. *Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra and let $E_{\mathcal{N}}$ be the trace preserving conditional expectation onto $\mathcal{N}$. Then, for any $b \in \mathcal{M}^+$,*

(i) $\|E_{\mathcal{N}}(b)\|_1 \leqslant \|b\|_1$.
(ii) $\overline{E_{\mathcal{N}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \subset \Theta_{\mathcal{N}}(b) \cap \mathcal{N}^+$.

Proof. (i) is proved in [3]. To see (ii) note that by Remark 2.2, for every $v \in \mathcal{M}$ such that $\|v\| \leqslant 1$, $v^* b v \precsim b$; by Theorem 2.2 in [3], $E_{\mathcal{N}}(v^* b v) \prec v^* b v$. So by transitivity $E_{\mathcal{N}}(v^* b v) \in$

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$\Theta_{\mathcal{N}}(b) \cap \mathcal{N}^+$. If $(a_n)_{n\in\mathbb{N}} \subset E_{\mathcal{N}}(\mathcal{C}_{\mathcal{M}}(b))$ is such that $\lim_{n\to\infty} \|a_n - a\|_1 = 0$ for some $a \in \mathcal{N}$, then necessarily $a \in \mathcal{N}^+$. By the previous argument we have that $a_n \prec_w b$ for every n. Therefore, by (1),

$$\int_0^s \lambda_a(t)\,dt = \lim_{n\to\infty} \int_0^s \lambda_{a_n}(t)\,dt \leqslant \int_0^s \lambda_b(t)\,dt$$

and so $a \prec_w b$. □

Next we prove our main result, which complements Theorem 2.4 in the case of submajorization and contractive orbits.

Theorem 3.4. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra of $\mathcal{M}$ and let $b \in \mathcal{M}^+$. Then*

$$\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} = \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+. \tag{4}$$

Proof. By (ii) in Lemma 3.3, $\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \subset \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+$. To prove the other inclusion, let $a \in \mathcal{A}^+$ be such that $a \prec_w b$. By (i) in Proposition 3.1 there exists $c \in \mathcal{M}^+$ such that $a \prec c \leqslant b$. By Theorem 2.4,

$$a \in \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(c))}^{\sigma\text{-sot}}. \tag{5}$$

Note that, since $c \leqslant b$, $c \precsim b$ (Remark 2.2). Thus, by (ii) in Proposition 3.1,

$$c \in \overline{\mathcal{C}_{\mathcal{M}}(b)}. \tag{6}$$

Let $\varepsilon > 0$. By (5) and (6) there exist $u \in \mathcal{U}_{\mathcal{M}}$ and a contraction $v \in \mathcal{M}$ such that $\|a - E_{\mathcal{A}}(u^*cu)\|_1 \leqslant \varepsilon$ and $\|c - v^*bv\| \leqslant \varepsilon$. Therefore,

$$\big\|E_{\mathcal{A}}(u^*cu) - E_{\mathcal{A}}((vu)^*b(vu))\big\|_1 = \big\|E_{\mathcal{A}}(u^*(c - v^*bv)u)\big\|_1 \leqslant \varepsilon,$$

since $\|x\|_1 \leqslant \|x\|$ and $E_{\mathcal{A}} \circ Ad_u$ is a $\|\cdot\|_1$-contraction (Lemma 3.3). Thus

$$\begin{aligned}\big\|a - E_{\mathcal{A}}((vu)^*b(vu))\big\|_1 &\leqslant \big\|a - E_{\mathcal{A}}(u^*cu)\big\|_1 \\ &\quad + \big\|E_{\mathcal{A}}(u^*cu) - E_{\mathcal{A}}((vu)^*b(vu))\big\|_1 \\ &\leqslant 2\varepsilon.\end{aligned}$$

As ε was arbitrary, we get $a \in \overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$, as desired. □

Corollary 3.5. *For each $b \in \mathcal{M}^+$, the set $\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$ is convex and σ-weakly compact.*

Proof. By (3) in Theorem 2.5 of [6],

$$\Theta_{\mathcal{M}}(b) = \overline{\mathrm{co}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}. \tag{7}$$

The right-hand side is bounded, convex, and σ-strongly closed, so it is σ-weakly closed and thus compact. Then

$$\Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+ = \Theta_{\mathcal{M}}(b) \cap \mathcal{A}^+ = \overline{\mathrm{co}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \cap \mathcal{A}^+$$

is convex and σ-weakly compact. By Theorem 3.4, we are done. □

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Remark 3.6. For any $b \in \mathcal{M}^+$, the property of $\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$ being convex is essentially equivalent to Theorem 3.4. Indeed, assuming $\overline{E_{\mathcal{A}}(\mathcal{C}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$ to be convex and using (7),

$$\begin{aligned}\Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+ &\subset E_{\mathcal{A}}\big(\overline{\mathrm{co}\big(\mathcal{C}_{\mathcal{M}}(b)\big)}^{\sigma\text{-sot}}\big) \subset \overline{E_{\mathcal{A}}\big(\mathrm{co}\big(\mathcal{C}_{\mathcal{M}}(b)\big)\big)}^{\sigma\text{-sot}} \\ &= \overline{\mathrm{co}\, E_{\mathcal{A}}\big(\big(\mathcal{C}_{\mathcal{M}}(b)\big)\big)}^{\sigma\text{-sot}} = \overline{E_{\mathcal{A}}\big(\mathcal{C}_{\mathcal{M}}(b)\big)}^{\sigma\text{-sot}},\end{aligned}$$

where we have used that $E_{\mathcal{A}}$ is $\|\cdot\|_1$-continuous (by (i) in Lemma 3.3). The reverse inclusion is given by (ii) in Lemma 3.3.

4. A reformulation of the Arveson–Kadison problem

Let $\mathcal{A} \subset \mathcal{M}$ be a masa and $b \in \mathcal{M}^+$. In [5], Arveson and Kadison pose the problem of whether

$$E_{\mathcal{A}}\big(\overline{\mathcal{U}_{\mathcal{M}}(b)}\big) = \Omega_{\mathcal{A}}(b). \tag{8}$$

Similarly, with regard to the results of the present paper, it is natural to ask whether

$$E_{\mathcal{A}}\big(\overline{\mathcal{C}_{\mathcal{M}}(b)}\big) = \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+. \tag{9}$$

It turns out that the two problems are equivalent, even in the broader class of diffuse abelian subalgebras.

Theorem 4.1. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian subalgebra. Then the following statements are equivalent*:

(i) $\forall b \in \mathcal{M}^{\mathrm{sa}}$, $E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}) = \Omega_{\mathcal{A}}(b)$;
(ii) $\forall b \in \mathcal{M}^+$, $E_{\mathcal{A}}(\overline{\mathcal{C}_{\mathcal{M}}(b)}) = \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+$.

Proof. Using arguments similar to those in Lemma 3.3 we can prove that for $b \in \mathcal{M}^{\mathrm{sa}}$, $E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}) \subset \Omega_{\mathcal{A}}(b)$. If $b \in \mathcal{M}^+$, using the norm-continuity of $E_{\mathcal{A}}$ and Lemma 3.3,

$$E_{\mathcal{A}}\big(\overline{\mathcal{C}_{\mathcal{M}}(b)}\big) \subset \overline{E_{\mathcal{A}}\big(\mathcal{C}_{\mathcal{M}}(b)\big)} \subset \overline{E_{\mathcal{A}}\big(\mathcal{C}_{\mathcal{M}}(b)\big)}^{\sigma\text{-sot}} \subset \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+.$$

(i) $\Rightarrow$ (ii). Let $a \in \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+$, $b \in \mathcal{M}^+$. Since $\mathcal{A}$ is diffuse and abelian, by Proposition 2.3 there exists an spectral resolution of the identity $\{e(t)\}_{t\in[0,1]} \subseteq \mathcal{A}$ such that $\tau(e(t)) = t$ for $t \in [0,1]$ and such that $a = \int_0^1 \lambda_a(t)\, de(t)$. Consider the operator $b' = \int_0^1 \lambda_b(t)\, de(t)$. It is straightforward to verify that $\lambda_{b'} = \lambda_b$ so that, by Theorem 2.1, $\overline{\mathcal{U}_{\mathcal{M}}(b)} = \overline{\mathcal{U}_{\mathcal{M}}(b')}$. From this last fact it follows that $\overline{\mathcal{C}_{\mathcal{M}}(b)} = \overline{\mathcal{C}_{\mathcal{M}}(b')}$, and so after replacing b by b' we can assume that $b \in \mathcal{A}$. By Proposition 3.1 there exists $c \in \mathcal{A}^+$ such that $a \prec c \leqslant b$ and by hypothesis we get $a \in E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(c)})$. Again by Proposition 3.1, since $c \leqslant b$ implies $c \precsim b$, we get $c \in \overline{\mathcal{C}_{\mathcal{M}}(b)}$. Then $\mathcal{U}_{\mathcal{M}}(c) \subset \overline{\mathcal{C}_{\mathcal{M}}(b)}$, so we have $a \in E_{\mathcal{A}}(\overline{\mathcal{C}_{\mathcal{M}}(b)})$.

(ii) $\Rightarrow$ (i). Let $b \in \mathcal{M}^{\mathrm{sa}}$, $a \in \Omega_{\mathcal{A}}(b)$. Since $\lambda_{b+\alpha I} = \lambda_b + \alpha$, then $a \prec b$ if and only if $a + \alpha \prec b + \alpha$. Hence, $\Omega_{\mathcal{A}}(b + \alpha I) = \Omega_{\mathcal{A}}(b) + \alpha I$, and it is clear that $E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b + \alpha I)}) = E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}) + \alpha I$. Thus, we can assume without loss of generality that $a, b \in \mathcal{M}^+$. The following argument was inspired by the proof of Theorem 4.1 in [5]. Since in particular $a \in \Theta_{\mathcal{A}}(b) \cap \mathcal{A}^+$, by hypothesis there exist $c \in \mathcal{M}^+$ and a sequence $(v_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ with $\|v_n\| \leqslant 1$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \big\|v_n^* b v_n - c\big\| = 0 \quad \text{and} \quad E_{\mathcal{A}}(c) = a. \tag{10}$$

So $\tau(c) = \tau(a) = \tau(b)$. Let $p = p^b(0, \|b\|]$.

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Claim. $\lim_{n\to\infty} \|p - |v_n^* p|\|_1 = 0$.

Since $\mathcal{M}$ is a finite factor, the partial isometry in the polar decomposition of $v_n^* p$ can be extended to a unitary u_n: so $v_n^* p = u_n |v_n^* p|$. Thus

$$\begin{aligned}\|v_n^* b v_n - u_n b u_n^*\|_1 &= \|v_n^* p b (v_n^* p)^* - u_n b u_n^*\|_1\\ &= \|u_n |v_n^* p| b |v_n^* p| u_n^* - u_n b u_n^*\|_1\\ &= \||v_n^* p| b |p v_n^*| - b\|_1\\ &\leqslant \|(|v_n^* p| - p) b |v_n^* p|\|_1 + \|b(|v_n^* p| - p)\|_1\\ &\leqslant \||v_n^* p| - p\|_1 \|b|v_n^* p|\| + \|b\| \||v_n^* p| - p\|_1\\ &\leqslant 2\|b\| \||v_n^* p| - p\|_1 \xrightarrow[n]{} 0.\end{aligned}$$

By (10) and the inequalities above, $\lim_n \|c - u_n b u_n^*\|_1 = 0$, and so $c \in \overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}$. Using Theorem 2.1,

$$a = E_{\mathcal{A}}(c) \in E_{\mathcal{A}}\big(\overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}\big) = E_{\mathcal{A}}\big(\overline{\mathcal{U}_{\mathcal{M}}(b)}\big).$$

Proof of the claim. Since $\|v_n\| \leqslant 1$, $p v_n v_n^* p \leqslant p$, and so $|v_n^* p| \leqslant p$. Then by (10),

$$\begin{aligned}0 &\leqslant \lim_n \tau\big((1 - v_n v_n^*) b\big) = \lim_n \tau\big(b - v_n^* b v_n\big)\\ &= \lim_n \tau\big(c - v_n^* b v_n\big) \leqslant \lim_n \|c - v_n^* b v_n\| = 0.\end{aligned}$$

Let $\varepsilon > 0$. Since $b(b+\delta)^{-1} \nearrow p$ strongly when $\delta \to 0$ and

$$0 \leqslant \tau\big((1 - v_n v_n^*)\big(p - b(b+\delta)^{-1}\big)\big) \leqslant \tau\big(p - b(b+\delta)^{-1}\big),$$

we can choose δ such that $\tau((1 - v_n v_n^*)p) \leqslant \varepsilon + \tau((1 - v_n v_n^*) b (b+\delta)^{-1})$ for every $n \in \mathbb{N}$. Then, choosing n such that $\tau((1 - v_n v_n^*) b) \leqslant \|(b+\delta)^{-1}\|^{-1} \varepsilon$, we obtain

$$\begin{aligned}0 &\leqslant \tau\big((1 - v_n v_n^*) p\big) \leqslant \varepsilon + \tau\big((1 - v_n v_n^*) b (b+\delta)^{-1}\big)\\ &= \varepsilon + \tau\big((1 - v_n v_n^*)^{1/2} b^{1/2} (b+\delta)^{-1} b^{1/2} (1 - v_n v_n^*)^{1/2}\big)\\ &\leqslant \varepsilon + \|(b+\delta)^{-1}\| \tau\big((1 - v_n v_n^*) b\big) \leqslant 2\varepsilon.\end{aligned}$$

Therefore, $\lim_n \tau((1 - v_n v_n^*) p) = 0$. For any $x \in \mathcal{M}^+$ with $\|x\| \leqslant 1$, $x - x^2 = x(1-x) = x^{1/2}(1-x)x^{1/2} \geqslant 0$. Since $\|v_n^* p\| \leqslant 1$, we conclude that $|v_n^* p|^2 \leqslant |v_n^* p|$, and so

$$\begin{aligned}\|p - |v_n^* p|\|_1 &= \tau\big(p - |v_n^* p|\big) \leqslant \tau\big(p - |v_n^* p|^2\big)\\ &= \tau\big(p - p v_n v_n^* p\big) = \tau\big((1 - v_n v_n^*) p\big) \to 0. \qquad \square\end{aligned}$$

We finish with the following remark concerning the relation between our main result and the problem (9). The characterization in Theorem 3.4 of the positive operators in a diffuse abelian subalgebra $\mathcal{A}$ majorized by a fixed $b \in \mathcal{M}^+$ is weaker than that posed in (9), since in general (using Corollary 3.2)

$$E_{\mathcal{A}}\big(\overline{\mathcal{C}_{\mathcal{M}}(b)}\big) = E_{\mathcal{A}}\big(\overline{\mathcal{C}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}\big) \subset \overline{E_{\mathcal{A}}\big(\mathcal{C}_{\mathcal{M}}(b)\big)}^{\sigma\text{-sot}}. \tag{11}$$

By Theorems 3.4 and 4.1, an affirmative answer to the Arveson–Kadison problem would imply equality in (11) and, conversely, equality in (11) would settle the Arveson–Kadison problem affirmatively.

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