

A Schur-Horn Theorem in II_1 Factors

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ABSTRACT. Given a II_1 factor \mathcal{M} and a diffuse abelian von Neumann subalgebra $\mathcal{A} \subset \mathcal{M}$, we prove a version of the Schur-Horn theorem, namely

$$\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} = \{a \in \mathcal{A}^{\text{sa}} : a \prec b\}, \quad b \in \mathcal{M}^{\text{sa}},$$

where \prec denotes spectral majorization, $E_{\mathcal{A}}$ the unique trace-preserving conditional expectation onto \mathcal{A} , and $\mathcal{U}_{\mathcal{M}}(b)$ the unitary orbit of b in \mathcal{M} . This result is inspired by a recent problem posed by Arveson and Kadison.

1. INTRODUCTION

In 1923, I. Schur [18] proved that if $A \in M_n(\mathbb{C})^{\text{sa}}$ (i.e., A is selfadjoint), then

$$\sum_{j=1}^k \alpha_j^\downarrow \leq \sum_{j=1}^k \beta_j^\downarrow, \quad k = 1, \dots, n,$$

with equality when $k = n$ (denoted $\alpha \prec \beta$), where $\alpha = \text{diag}(A) \in \mathbb{R}^n$, $\beta = \lambda(A) \in \mathbb{R}^n$ the spectrum (counting multiplicity) of A , and $\alpha^\downarrow, \beta^\downarrow \in \mathbb{R}^n$ are obtained from α, β by reordering their entries in decreasing order.

In 1954, A. Horn [12] proved the converse: given $\alpha, \beta \in \mathbb{R}^n$ with $\alpha \prec \beta$, there exists $A \in M_n(\mathbb{C})^{\text{sa}}$ such that $\text{diag}(A) = \alpha$, $\lambda(A) = \beta$. Since every selfadjoint matrix is diagonalizable, the results of Schur and Horn can be combined in the following assertion: if \mathcal{D} denotes the diagonal masa in $M_n(\mathbb{C})$ and $E_{\mathcal{D}}$ is the compression onto \mathcal{D} , then

$$(1.1) \quad E_{\mathcal{D}}(\{U M_{\beta} U^* : U \in M_n(\mathbb{C}) \text{ unitary}\}) = \{M_{\alpha} \in \mathcal{D} : \alpha \prec \beta\},$$

where M_{α} is the diagonal matrix with the entries of α in the main diagonal.

This combination of the two results, commonly known as Schur–Horn theorem, has played a significant role in many contexts of matrix analysis: although simple, vector majorization expresses a natural and deep relation among vectors, and as such it has been a useful tool both in pure and applied mathematics. We refer to the books [5, 14] and the introductions of [6, 16] for more on this.

During the last 25 years, several extensions of majorization have been proposed by, among others, Ando [1] (to selfadjoint matrices), Kamei [13] (to selfadjoint operators in a II_1 factor), Hiai [10, 11] (to normal operators in a von Neumann algebra), and Neumann [16] (to vectors in $\ell^\infty(\mathbb{N})$). With these generalizations at hand, it is natural to ask about extensions of the Schur–Horn theorem.

In [16], Neumann developed his extension of majorization with the goal of using it to prove a Schur–Horn type theorem in $\mathcal{B}(\mathcal{H})$ in the vein of previous works in convexity (see the introduction in [16] for details and bibliography). Other versions of the Schur–Horn theorem have been considered in [3] and [6]. It is interesting to note that the motivation in [16] comes from geometry, in [3] comes from the study of frames on Hilbert spaces, while in [6] it is of an operator theoretic nature.

In [6] Arveson and Kadison proposed the study of a Schur–Horn type theorem in the context of II_1 factors, which are for such purpose the most natural generalization of full matrix algebras. They proved a Schur type theorem for II_1 factors and they posed as a problem a converse of this result, i.e., a Horn type theorem. In this note we prove a Schur–Horn type theorem that is inspired by Arveson–Kadison’s conjecture (Theorem 3.4).

2. PRELIMINARIES

Throughout the paper \mathcal{M} denotes a II_1 factor with normalized faithful normal trace τ . We denote by \mathcal{M}^{sa} , \mathcal{M}^+ , $\mathcal{U}_{\mathcal{M}}$, the sets of selfadjoint, positive, and unitary elements of \mathcal{M} , and by $\mathcal{Z}(\mathcal{N})$ the centre of a von Neumann algebra \mathcal{N} . Given $a \in \mathcal{M}^{\text{sa}}$ we denote its spectral measure by p^a . The characteristic function of the set Δ is denoted by 1_{Δ} . For $n \in \mathbb{N}$, the algebra of $n \times n$ matrices over \mathbb{C} is denoted by $M_n(\mathbb{C})$, and its unitary group by \mathcal{U}_n . By dt we denote integration with respect to Lebesgue measure. To simplify terminology, we will refer to non-decreasing functions simply as “increasing”; similarly, “decreasing” will be used instead of “non-increasing”.

Besides the usual operator norm in \mathcal{M} , we consider the 1-norm induced by the trace, $\|x\|_1 = \tau(|x|)$. As we will be always dealing with bounded sets in a II_1 factor, we can profit from the fact that the topology induced by $\|\cdot\|_1$ agrees with the σ -strong operator topology. Because of this we will express our results in terms of σ -strong closures although our computations are based on estimates for $\|\cdot\|_1$. For $X \subset \mathcal{M}$, we shall denote by \bar{X} and $\bar{X}^{\sigma\text{-sot}}$ the respective closures in the norm and in the σ -strong operator topology.

2.1. Spectral scale and spectral preorders. The *spectral scale* [17] of $a \in \mathcal{M}^{\text{sa}}$ is defined as

$$\lambda_a(t) = \min \left\{ s \in \mathbb{R} : \tau(p^a(s, \infty)) \leq t \right\}, \quad t \in [0, 1].$$

The function $\lambda_a : [0, 1] \rightarrow [0, \|a\|]$ is decreasing and right-continuous. The map $a \mapsto \lambda_a$ is continuous with respect to both $\|\cdot\|$ and $\|\cdot\|_1$, since [17]

$$(2.1) \quad \|\lambda_a - \lambda_b\|_\infty \leq \|a - b\|, \quad \|\lambda_a - \lambda_b\|_1 \leq \|a - b\|_1 \quad a, b \in \mathcal{M}^{\text{sa}},$$

where the norms on the left are those of $L^\infty([0, 1], dt)$ and $L^1([0, 1], dt)$ respectively.

We say that a is *submajorized* by b , written $a \prec_w b$, if

$$\int_0^s \lambda_a(t) dt \leq \int_0^s \lambda_b(t) dt, \quad \text{for every } s \in [0, 1].$$

If in addition $\tau(a) = \tau(b)$ then we say that a is *majorized* by b , written $a \prec b$.

These preorders play an important role in many papers (among them we mention [8, 10, 11, 13]), and they arise naturally in several contexts in operator theory and operator algebras: some recent examples closely related to our work are the study of Young’s type [9] and Jensen’s type inequalities [2, 6, 7, 10].

Theorem 2.1 ([10]). *Let $a, b \in \mathcal{M}^{\text{sa}}$. Then $a \prec b$ (resp. $a \prec_w b$) if and only if $\tau(f(a)) \leq \tau(f(b))$ for every convex (resp. increasing convex) function $f : J \rightarrow \mathbb{R}$, where J is an open interval such that $\sigma(a), \sigma(b) \subseteq J$.*

If $\mathcal{N} \subset \mathcal{M}$ is a von Neumann subalgebra and $b \in \mathcal{M}^{\text{sa}}$, we denote by $\Omega_{\mathcal{N}}(b)$ the set of elements in \mathcal{N}^{sa} that are majorized by b , i.e.,

$$\Omega_{\mathcal{N}}(b) = \{a \in \mathcal{N}^{\text{sa}} : a \prec b\}.$$

The unitary orbit of $a \in \mathcal{M}^{\text{sa}}$ is the set $\mathcal{U}_{\mathcal{M}}(a) = \{u^* a u : u \in \mathcal{U}_{\mathcal{M}}\}$.

Proposition 2.2. *Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra and let $E_{\mathcal{N}}$ be the trace preserving conditional expectation onto \mathcal{N} . Then, for any $b \in \mathcal{M}^{\text{sa}}$,*

- (i) $E_{\mathcal{N}}(b) \prec b$.
- (ii) $\|E_{\mathcal{N}}(b)\|_1 \leq \|b\|_1$.
- (iii) $\overline{E_{\mathcal{N}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \subset \Omega_{\mathcal{N}}(b)$.

Proof. (i) The map $E_{\mathcal{N}}$ is doubly stochastic (i.e., trace preserving, unital, and positive), so this follows from [10, Theorem 4.7] (see also [6, Theorem 7.2]).

- (ii) Consider the convex function $f(x) = |x|$. Since $E_{\mathcal{N}}(b) \prec b$, using Theorem 2.1 we get

$$\|E_{\mathcal{N}}(b)\|_1 = \tau(f(E_{\mathcal{N}}(b))) \leq \tau(f(b)) = \|b\|_1.$$

- (iii) By (i) and the fact that $ubu^* < b$ for every $u \in \mathcal{U}_{\mathcal{M}}$, we just have to prove that the set $\Omega_{\mathcal{N}}(b)$ is $\|\cdot\|_1$ -closed. So, let $(a_n)_{n \in \mathbb{N}} \subset \Omega_{\mathcal{N}}(b)$ be such that $\lim_{n \rightarrow \infty} \|a_n - a\|_1 = 0$ for some $a \in \mathcal{N}$. Then, necessarily, $a \in \mathcal{N}^{sa}$. By (2.1),

$$\int_0^s \lambda_a(t) dt = \lim_{n \rightarrow \infty} \int_0^s \lambda_{a_n}(t) dt \leq \int_0^s \lambda_b(t) dt.$$

Also, $\tau(a) = \lim_n \tau(a_n) = \tau(b)$, so $a < b$. □

The following result seems to be well-known, but we have not been able to find a reference. Thus we give a sketch of a proof.

Proposition 2.3. *Let $a \in \mathcal{A}^{sa}$, where $\mathcal{A} \subset \mathcal{M}$ is a diffuse von Neumann subalgebra. Then there exists a spectral resolution $\{e(t)\}_{t \in [0,1]} \subset \mathcal{A}$ with $\tau(e(t)) = t$ for every $t \in [0, 1]$, and such that*

$$a = \int_0^1 \lambda_a(t) de(t).$$

Proof. Since $\tau(1) < \infty$, it is enough to show the result for $a \geq 0$. By [15, Theorem 3.2] there exists $a' \in \mathcal{A}$ with $g_{a'}(s) = \tau(p^{a'}(-\infty, s])$ continuous in \mathbb{R} , and an increasing left-continuous function f such that $p^a(-\infty, s] = p^{a'}(-\infty, f(s)]$. Although the original statement in [15] involves a masa, only the fact that the algebra is diffuse is needed for its proof.

Let $g_{a'}^\dagger(t) = \min\{s : g_{a'}(s) \geq t\}$, and let $q(t) = p^{a'}(-\infty, g_{a'}^\dagger(t)]$. Since $\tau(q(t)) = g_{a'}(g_{a'}^\dagger(t)) = t$ for every $t \in [0, 1]$, it follows that $\{q(t)\}_{t \in [0,1]}$ is a continuous spectral resolution. Moreover, $q(g_{a'}(t)) = p^{a'}(-\infty, t]$, so

$$p^a(-\infty, t] = q(g_{a'}(f(t))).$$

As $g_{a'} \circ f$ is increasing and right-continuous, by [15, Theorem 4.4] there exists an increasing and left-continuous function h_a such that

$$a = \int h_a(t) dq(t).$$

Define $e(t) = 1 - q(1 - t)$, and $h(t) = h_a(1 - t)$. Then $\tau(e(t)) = t$,

$$a = \int h(t) de(t).$$

As h is decreasing and right-continuous, it can be seen that $h = \lambda_a$. □

A spectral resolution $\{e(t)\}_{t \in [0,1]}$ as in Proposition 2.3 is called a *complete flag* for a .

3. A SCHUR-HORN THEOREM FOR II_1 FACTORS

For each $n \in \mathbb{N}$, $k \in 1, \dots, 2^n$, let $\{I_k^{(n)}\}_{k=1}^{2^n}$ be the partition of $[0, 1]$ associated to the points $\{h 2^{-n} : h = 0, \dots, 2^n\}$.

Definition 3.1. For each $n \in \mathbb{N}$ and every $f \in L^1([0, 1])$, let

$$E_n(f) = \sum_{i=1}^{2^n} \left(2^n \int_{I_i^{(n)}} f \right) 1_{I_i^{(n)}}.$$

It is clear that each operator E_n is a linear contraction for both $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

Given the flag $\{e(t)\}_t$, we write $e([t_0, t_1])$ for $e(t_1) - e(t_0)$. Note that, since we consider $e(t)$ diffuse, $e([t_0, t_1]) = e((t_0, t_1)) = e([t_0, t_1)) = e((t_0, t_1])$.

Lemma 3.2. Let $\{e(t)\}_{t \in [0,1]}$, $\{I_i^{(n)}\}_{i=1}^{2^n}$, $n \in \mathbb{N}$ and $\{E_n\}_{n \in \mathbb{N}}$ as above. Then, for each $a \in \mathcal{M}^{sa}$,

$$(3.1) \quad \lim_{n \rightarrow \infty} \left\| a - \int_0^1 E_n(\lambda_a)(t) de(t) \right\|_1 = 0.$$

Proof. By continuity of the trace, we only need to check that

$$(3.2) \quad \|\lambda_a - E_n(\lambda_a)\|_1 \xrightarrow{n} 0$$

in $L^1([0, 1])$. Consider first a continuous function g . By uniform continuity, $\|g - E_n(g)\|_1 \rightarrow 0$. Since continuous functions are dense in $L^1([0, 1])$ and because the operators E_n are $\|\cdot\|_1$ -contractive for every $n \in \mathbb{N}$, a standard $\varepsilon/3$ argument proves (3.2) for any integrable function. \square

Recall that \mathcal{D} denotes the diagonal masa in $M_n(\mathbb{C})$, and that for $\alpha \in \mathbb{R}^n$ we denote by M_α the matrix with the entries of α in the diagonal and zero off-diagonal. The projection $E_{\mathcal{D}}$ of $M_n(\mathbb{C})$ onto \mathcal{D} is then given by $E_{\mathcal{D}}(A) = M_{\text{diag}(A)}$, where $\text{diag}(A) \in \mathbb{R}^n$ is the main diagonal of A . We use $\{e_{ij}\}$ to denote the canonical system of matrix units in $M_n(\mathbb{C})$.

Lemma 3.3. Let $\mathcal{N} \subset \mathcal{M}$ be a von Neumann subalgebra, and assume $E_{\mathcal{N}}$ denotes the unique trace preserving conditional expectation onto \mathcal{N} . Let $\{p_i\}_{i=1}^n \subset \mathcal{Z}(\mathcal{N})$ be a set of mutually orthogonal equivalent projections such that $\sum_{i=1}^n p_i = I$. Then there exists a unital $*$ -monomorphism $\pi : M_n(\mathbb{C}) \rightarrow \mathcal{M}$ satisfying

$$(3.3) \quad \pi(e_{ii}) = p_i, \quad 1 \leq i \leq n,$$

$$(3.4) \quad E_{\mathcal{N}}(\pi(A)) = \pi(E_{\mathcal{D}}(A)), \quad A \in M_n(\mathbb{C}).$$

Proof. Since the projections p_i are equivalent in \mathcal{M} , for each i there exists a partial isometry v_{i1} such that $v_{i1}v_{i1}^* = p_i$ and $v_{i1}^*v_{i1} = p_1$. Let $v_{11} = p_1$, $v_{1i} = v_{i1}^*$ for $2 \leq i \leq n$, and $v_{ij} = v_{i1}v_{1j}$ for $1 \leq i, j \leq n$. In this way we get the standard associated system of matrix units $\{v_{ij}, 1 \leq i, j \leq n\}$ in \mathcal{M} . Define $\pi : M_n(\mathbb{C}) \rightarrow \mathcal{M}$ by $\pi(A) = \sum_{i,j=1}^n a_{ij}v_{ij}$.

The matrix unit relations imply that π is a $*$ -monomorphism and it is clear that (3.3) is also satisfied. Moreover,

$$E_{\mathcal{N}}(v_{ij}) = E_{\mathcal{N}}(p_i v_{ij} p_j) = p_i E_{\mathcal{N}}(v_{ij}) p_j = \delta_{ij} p_i,$$

since $p_i p_j = \delta_{ij} p_i$, $E_{\mathcal{N}}(v_{ii}) = E_{\mathcal{N}}(p_i) = p_i$, and $p_i \in \mathcal{Z}(\mathcal{N})$. Finally, we check (3.4):

$$E_{\mathcal{N}}(\pi(A)) = \sum_{i,j} a_{ij} E_{\mathcal{N}}(v_{ij}) = \sum_i a_{ii} p_i = \pi(E_D(A)). \quad \square$$

Next we state and prove our version of the Schur-Horn theorem for II_1 factors. Note the formal analogy with (1.1).

Theorem 3.4. *Let $\mathcal{A} \subset \mathcal{M}$ be a diffuse abelian von Neumann subalgebra and let $b \in \mathcal{M}^{\text{sa}}$. Then*

$$(3.5) \quad \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} = \Omega_{\mathcal{A}}(b).$$

Proof. By Proposition 2.2, we only need to prove $\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}} \supset \Omega_{\mathcal{A}}(b)$. So let $a \in \mathcal{A}^{\text{sa}}$ with $a < b$. By Proposition 2.3,

$$a = \int_0^1 \lambda_a(t) de(t), \quad b = \int_0^1 \lambda_b(t) df(t)$$

where $\{e(t)\}$ and $\{f(t)\}$ are complete flags with $\tau(e(t)) = \tau(f(t)) = t$, $e(t) \in \mathcal{A}$, $t \in [0, 1]$.

Let $\{I_i^{(n)}\}_{i=1}^{2^n}$, $n \in \mathbb{N}$, be the family of partitions considered before and let $\varepsilon > 0$. By Lemma 3.2 there exists $n \in \mathbb{N}$ such that

$$(3.6) \quad \left\| a - \sum_{i=1}^{2^n} \alpha_i p_i \right\|_1 < \varepsilon, \quad \left\| b - \sum_{i=1}^{2^n} \beta_i q_i \right\|_1 < \varepsilon,$$

where $\alpha_i = 2^n \int_{I_i^{(n)}} \lambda_a(t) dt$, $\beta_i = 2^n \int_{I_i^{(n)}} \lambda_b(t) dt$, $p_i = e(I_i^{(n)})$, $q_i = f(I_i^{(n)})$, $1 \leq i \leq 2^n$. Note that $\tau(p_i) = \tau(q_i) = 2^{-n}$. Let $\alpha = (\alpha_1, \dots, \alpha_{2^n})$, $\beta = (\beta_1, \dots, \beta_{2^n}) \in \mathbb{R}^{2^n}$. From the fact that λ_a and λ_b are decreasing, the entries of α and β are already in decreasing order. Using that $a < b$ in \mathcal{M} , we conclude that $\alpha < \beta$ in \mathbb{R}^n .

By the classical Schur-Horn theorem (1.1), there exists $U \in \mathcal{U}_n(\mathbb{C})$ such that

$$(3.7) \quad E_{\mathcal{D}}(UM_{\beta}U^*) = M_{\alpha}.$$

Consider the $*$ -monomorphism π of Lemma 3.3 associated with the orthogonal family of projections $\{p_i\}_{i=1}^{2^n} \subset \mathcal{A}$. Let $w \in \mathcal{U}_{\mathcal{M}}$ such that $wq_iw^* = p_i$, $i = 1, \dots, 2^n$, and put $u := \pi(U)w \in \mathcal{U}_{\mathcal{M}}$. By (3.4) and (3.7),

$$\begin{aligned} E_{\mathcal{A}}\left(u\left(\sum_{i=1}^{2^n} \beta_i q_i\right)u^*\right) &= E_{\mathcal{A}}(\pi(UM_{\beta}U^*)) \\ &= \pi(E_{\mathcal{D}}(UM_{\beta}U^*)) = \pi(M_{\alpha}) = \sum_{i=1}^{2^n} \alpha_i p_i. \end{aligned}$$

Using (3.6), we conclude that

$$\|E_{\mathcal{A}}(ubu^*) - a\|_1 \leq \left\|E_{\mathcal{A}}\left(u\left(b - \sum_{i=1}^{2^n} \beta_i q_i\right)u^*\right)\right\|_1 + \left\|a - \sum_{i=1}^{2^n} \alpha_i p_i\right\|_1 < 2\varepsilon.$$

As ε was arbitrary, we obtain $a \in \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$. □

Corollary 3.5. *For each $b \in \mathcal{M}^+$, the set $\overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}$ is convex and σ -weakly compact.*

In [6], Arveson and Kadison posed the problem whether for $b \in \mathcal{M}^{\text{sa}}$, with the notations of Theorem 3.4,

$$E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}) = \Omega_{\mathcal{A}}(b).$$

Since ([13])

$$\overline{\mathcal{U}_{\mathcal{M}}(b)} = \overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}} = \{a \in \mathcal{M}^{\text{sa}} : \lambda_a = \lambda_b\},$$

an affirmative answer to the Arveson-Kadison problem is equivalent to

$$(3.8) \quad E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}) = \Omega_{\mathcal{A}}(b).$$

As a description of the set $\Omega_{\mathcal{A}}(b)$, (3.5) is weaker than (3.8), since in general

$$(3.9) \quad E_{\mathcal{A}}(\overline{\mathcal{U}_{\mathcal{M}}(b)}^{\sigma\text{-sot}}) \subset \overline{E_{\mathcal{A}}(\mathcal{U}_{\mathcal{M}}(b))}^{\sigma\text{-sot}}.$$

An affirmative answer to the Arveson-Kadison problem would imply equality in (3.9). We think it is indeed the case, although a proof of this does not emerge from our present methods.

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