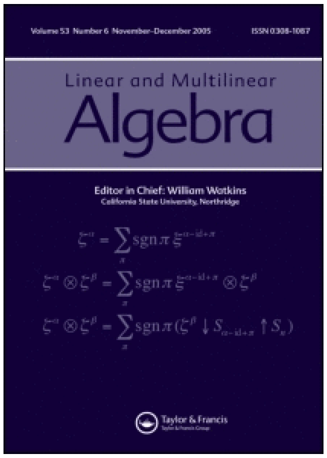

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On Tate's trace

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The problem of whether Tate's trace is linear or not is reduced to a special case.

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1. Introduction

In 1968, J. Tate [2] extended the usual notion of trace of an endomorphism to include the case where the underlying vector space is infinite dimensional, provided the given endomorphism is "finite potent", which, as its name indicates, means that some power of the endomorphism has finite rank.

While Tate's trace enjoys many of the properties of the classical trace, it is still an open problem whether this trace is linear, i.e. whether

$$\mathrm{Tr}_V(\varphi_1 + \varphi_2) = \mathrm{Tr}_V(\varphi_1) + \mathrm{Tr}_V(\varphi_2)$$

holds whenever φ_1, φ_2 and $\varphi_1 + \varphi_2$ are finite potent endomorphisms of an arbitrary vector space V.

Here we take a closer look at finite potent endomorphisms and use this perspective to show that if a counterexample to linearity exists at all it must be of a special type.

Recently Pablos Romo [1] furnished a counterexample to the linearity of Tate's trace for *three* endomorphisms. As an indication of how great the leap is from three to two

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endomorphisms, we show that his set-up, namely the use of nilpotent endomorphisms of class two, will never yield a counterexample to Tate's original question.

At the end, we exhibit an example of a finite potent endomorphism with non-zero trace but whose matrix has all zero diagonal entries.

2. Finite potent endomorphisms

Let F be a field and let V be a vector space over F. Following J. Tate, an endomorphism φ of V is finite potent if $\varphi^m(V)$ is finite dimensional for some $m \in \mathbf{N}$.

Let φ be an endomorphism of V, and let $F[x]$ stand for the algebra of polynomials in the variable x with coefficients in F. We may view V as an $F[x]$-module via φ. Let

$$V_0 = \{v \in V \mid x^m v = 0 \text{ for some } m\},$$

$$V_1 = \{v \in V \mid f(x)v = 0 \text{ for some } f(x) \in F[x] \text{ relatively prime to } x\}.$$

These are φ-invariant subspaces of V satisfying $V_0 \cap V_1 = (0)$.

Proposition 2.1 *If φ is finite potent then $V = V_0 \oplus V_1$, where φ restricted to V_0 is nilpotent, V_1 is finite dimensional, and $\varphi|_{V_1} : V_1 \to V_1$ is an isomorphism.*

Conversely, if V admits a φ-invariant decomposition $V = U \oplus W$, such that $\varphi|_U$ is nilpotent, W finite dimensional, and $\varphi|_W : W \to W$ is an isomorphism, then φ is finite potent, $U = V_0$ and $W = V_1$.

Proof Suppose first that φ is finite potent. Then there exists $m \in \mathbf{N}$ such that $x^m V = W$ is finite dimensional. Since W is a finite dimensional φ-invariant subspace of V, we have $f(x)W = (0)$ for some non-zero $f(x) \in F[x]$. Therefore, $f(x)x^m V = (0)$, with $f(x)x^m \neq 0$, which shows that V is a torsion $F[x]$-module. As such, we have $V = V_0 \oplus V_1$. Now $\varphi^m V_0$ is included in V_0 and finite dimensional. It easily follows that φ restricted to $\varphi^m V_0$ is nilpotent, i.e. $\varphi^n \varphi^m V_0 = (0)$ for some n, which shows that φ restricted to V_0 is nilpotent. Also φ restricted to V_1 is injective, since $V_0 \cap V_1 = (0)$. But $\varphi^m V_1$ is a subspace of V_1 and finite dimensional. Therefore, $\varphi^m V_1 = V_1$ and V_1 is finite dimensional.

Suppose next V admits a decomposition $V = U \oplus W$ with the stated properties. Then $\varphi^m U = (0)$ for some m, so $\varphi^m V = \varphi^m W = W$ is finite dimensional. This shows that φ is finite potent. We finally prove that in such decomposition U and W are uniquely determined as being $U = V_0$ and $W = V_1$. Indeed, if $v \in V_1$ then $v = u + w$, where $u \in U$ and $w \in W$. Choose k large enough so that $\varphi^k u = 0$. Then $\varphi^k v = \varphi^k w \in W$. Let $f(x)$ be the minimal polynomial of φ restricted to V_1. Since φ is injective on V_1, $f(x)$ and x are relatively prime. Thus, there exist $a(x), b(x) \in F[x]$ such $1 = a(x)f(x) + b(x)x^k$. Applying this to v, we see that $v = b(x)x^k v \in W$. This shows that $V_1 \subseteq W$. The argument is reversible, so $V_1 = W$. By definition, we also have $U \subseteq V_0$. Since $V = U \oplus V_1$ there is no other choice but that $U = V_0$. ∎

Thus, an endomorphism is finite potent if and only if it is the direct sum of a nilpotent endomorphism and an isomorphism defined on a finite dimensional subspace.

3. Tate's trace

Let φ be a finite potent endomorphism with corresponding decomposition $V = V_0 \oplus V_1$. We use this decomposition to define a trace, which satisfies the three defining properties of Tate's trace. Let

$$\mathrm{Tr}_V(\varphi) = \mathrm{Tr}_{V_1}(\varphi), \tag{1}$$

where on the right hand side the usual trace is considered. It is clear that

(T1) If V is finite dimensional, then $\mathrm{Tr}_V(\varphi)$ is the ordinary trace.
(T2) If φ is nilpotent, then $\mathrm{Tr}_V(\varphi) = 0$.

The third defining property of Tate's trace is the following.

(T3) If W is a φ-invariant subspace of V, then $\mathrm{Tr}_V(\varphi) = \mathrm{Tr}_W(\varphi) + \mathrm{Tr}_{V/W}(\varphi)$.

To see this suppose first that $V = U \oplus W$ is a φ-invariant decomposition of V, where φ restricted to U is nilpotent (in fact, a great deal of these considerations goes through if $\varphi|_U$ is just locally nilpotent). The argument used in the proof of the Proposition 2.1 shows that $V_1 \subseteq W$, so that $W = (W \cap V_0) \oplus V_1$ is the canonical decomposition of W relative to φ. Thus by (1)

$$\mathrm{Tr}_V(\varphi) = \mathrm{Tr}_{V_1}(\varphi) = \mathrm{Tr}_W(\varphi). \tag{2}$$

Suppose next that W is merely a φ-invariant subspace of V. The decomposition of W relative to φ is easily seen to be

$$W = (W \cap V_0) \oplus (W \cap V_1),$$

so by (1)

$$\mathrm{Tr}_W(\varphi) = \mathrm{Tr}_{W \cap V_1}(\varphi). \tag{3}$$

Now as $F[x]$-modules we have

$$\frac{V}{W} = \frac{V_0 \oplus V_1}{(W \cap V_0) \oplus (W \cap V_1)} \cong \frac{V_0}{W \cap V_0} \oplus \frac{V_1}{W \cap V_1}.$$

As φ restricted to V_0 is nilpotent, the induced map on $V_0/(W \cap V_0)$ is also nilpotent. Thus by (2)

$$\mathrm{Tr}_{V/W}(\varphi) = \mathrm{Tr}_{V_1/(W \cap V_1)}(\varphi). \tag{4}$$

From the finite dimensional case we know that

$$\mathrm{Tr}_{V_1}(\varphi) = \mathrm{Tr}_{W \cap V_1}(\varphi) + \mathrm{Tr}_{V_1/(W \cap V_1)}(\varphi). \tag{5}$$

Combining equations (2)–(5) we obtain (T3).

Tate defines a subspace E of $\mathrm{End}(V)$ to be finite potent, if there exists $n \in \mathbf{N}$ such that the product of any n elements of E has finite rank. He then shows his trace to be linear when restricted to a finite potent subspace. In particular, if φ is finite potent and θ has finite rank then $E = \langle \varphi, \theta \rangle$ is easily seen to be finite potent, so in this case

$$\mathrm{Tr}_V(\varphi + \theta) = \mathrm{Tr}_V(\varphi) + \mathrm{Tr}_V(\theta).$$

Suppose next that φ_1 and φ_2 are finite potent endomorphisms of V such that $\varphi_1 + \varphi_2$ is also finite potent. The above description of a general finite potent endomorphism yields $\varphi_1 = \nu_1 + \theta_1$ and $\varphi_2 = \nu_2 + \theta_2$, where ν_1, ν_2 are nilpotent and θ_1, θ_2 have finite rank. We easily see that $\theta_1 + \theta_2$ has finite rank, $\nu_1 + \nu_2$ is finite potent and

$$\mathrm{Tr}_V(\varphi_1 + \varphi_2) = \mathrm{Tr}_V(\varphi_1) + \mathrm{Tr}_V(\varphi_2) \quad \text{if and only if } \mathrm{Tr}_V(\nu_1 + \nu_2) = \mathrm{Tr}_V(\nu_1) + \mathrm{Tr}_V(\nu_2).$$

This justifies Tate's reduction of his question to the following: if the sum of two nilpotent endomorphisms is finite potent, is this sum necessarily traceless?

4. Reduction to cyclic modules

Suppose henceforth that φ_1 and φ_1 are nilpotent with $\phi = \varphi_1 + \varphi_2$ finite potent. Let $F\langle s, t\rangle$ stand for the algebra of polynomials in the non-commuting variables s and t. We may view V as a left $F\langle s, t\rangle$-module by letting s act via φ_1 and t via φ_2. From this perspective, a subspace of V is an $F\langle s, t\rangle$-submodule if and only if it is invariant under φ_1 and φ_2, in which case it is also invariant under ϕ.

Let $V = V_0 \oplus V_1$ be the canonical decomposition of V relative to ϕ. Let $f(x) \in F[x]$ be the minimal polynomial of ϕ. We have $f(x) = x^l g(x)$ where $g(x)$ is monic and relatively prime to x. If $l = 0$ then $V = V_1$ is finite dimensional; if $l = 1$ then ϕ has finite rank; if $g(x) = 1$ then ϕ is nilpotent. These three cases can be excluded from further consideration. In any case we have $V_1 = \phi^l(V)$. If $\varphi_1^n = 0 = \varphi_2^m$, we may also assume that $n, m \geq 2$.

We know that V_1 is spanned by finitely many vectors, say $v_1, \ldots, v_n$. For $1 \leq i \leq n$ let W_i be the smallest subspace of V which contains v_i and is invariant under φ_1 and φ_2; in other words, W_i is the $F\langle s, t\rangle$-submodule of V generated by v_i. Consider the $F\langle s, t\rangle$-submodule $W = W_1 + \cdots + W_n$ of V. We have $\phi^l(V) = V_1$ and by construction $V_1 \subseteq W$; therefore, $\phi^l(V) \subseteq W$. Thus ϕ induces a nilpotent endomorphism on V/W; obviously so do φ_1 and φ_2. It follows from (T3) that

$$\mathrm{Tr}_V(\varphi_1) = \mathrm{Tr}_W(\varphi_1),\ \mathrm{Tr}_V(\varphi_2) = \mathrm{Tr}_W(\varphi_2),\ \mathrm{Tr}_V(\phi) = \mathrm{Tr}_W(\phi).$$

This reduces Tate's question to the space W, which has countable dimension. Furthermore, consider the sequence of $F\langle s, t\rangle$-submodules of W:

$$(0) \subseteq W_1 \subseteq W_1 + W_2 \subseteq \cdots \subseteq W_1 + \cdots + W_n = W,$$

with corresponding quotient modules:

$$Y_1 = W_1/(0),\ Y_2 = (W_1 + W_2)/W_1, \ldots, Y_n = W/(W_1 + \cdots + W_{n-1}).$$

Repeated application of (T3) shows that Tate's trace is linear on W if and only if it is linear on each $F\langle s,t\rangle$-module Y_i. By construction each W_i is a cyclic $F\langle s,t\rangle$-module, and therefore so is each Y_i. This reduces Tate's question to the case of a cyclic $F\langle s,t\rangle$-module $Y = F\langle s,t\rangle y$.

The map $F\langle s,t\rangle \to Y$ given by $r \mapsto ry$ for $r \in F\langle s,t\rangle$ is an epimorphism of left $F\langle s,t\rangle$-modules, whose kernel is a left ideal, say I. As φ_1 and φ_2 are nilpotent, we see that I contains the two sided ideal generated by s^n, t^m and $f(s+t)$ for some $n, m \geq 2$. Note that Y is isomorphic to $F\langle s,t\rangle/I$. We summarize these findings below.

THEOREM 4.1 *Tate's trace is linear if and only if it is linear for the following countable space and operators. The vector space is $V = F\langle s,t\rangle/I$, where I is a left ideal of $F\langle s,t\rangle$. The endomorphism φ_1 is the map $r + I \mapsto sr + I$, φ_2 is the map $r + I \mapsto tr + I$, and naturally ϕ is the map $r + I \mapsto (s+t)r + I$, for $r \in F\langle s,t\rangle$. The left ideal I must be chosen so that:*

(a) *I contains the two sided ideal generated by s^n, t^m, $f(s+t)$, for some $n, m \geq 2$ and $f(x) \in F[x]$, where $f(x) = x^l g(x)$, $l \geq 2$ and $g(x) \neq 1$ is monic and relatively prime to x.*
(b) *If J is the right ideal of $F\langle s,t\rangle$ generated by $(s+t)^l$, then $V_1 = (J+I)/I$ is finite dimensional and non-trivial.*
(c) *V is infinite dimensional.*

If a counterexample to Tate's linearity exists at all, it must necessarily be of the above type, and a suitable computing system should allow for attainable examples. For instance, if I satisfied (a) and (b) for some $g(x) = (x-a)^k$ with $(s+t)^l \notin I$ and $\mathrm{char}(F) = 0$, then $\mathrm{Tr}_V(\phi) \neq 0$. So far our efforts have been in vain with the systems at our disposal. We next give an application which relates to Pablos Romo's example.

PROPOSITION 4.2 *Let V be a vector space over F, and let φ_1 and φ_2 be finite potent endomorphisms of V such that $\phi = \varphi_1 + \varphi_2$ is also finite potent. Suppose these endomorphisms have respective minimal polynomials $f_1(x) = x^m g_1(x)$, $f_2(x) = x^n g_2(x)$ and $f(x) = x^l g(x)$, where $g_1(x), g_2(x)$ and $g(x)$ are relatively prime to x, and at least two of the exponents m, n, l do not exceed 2. Then $\mathrm{Tr}_V(\varphi_1 + \varphi_2) = \mathrm{Tr}_V(\varphi_1) + \mathrm{Tr}_V(\varphi_2)$.*

Proof By shuffling finite rank operators around and eliminating trivial cases, we are left with only one case to consider, namely the one in Theorem 4.1 with $n = 2 = m$. Thus, ϕ is left multiplication by $s+t$ modulo I, where I contains the two sided ideal (s^2, t^2). For $i \in \mathbf{N}$, let $a_i = \underbrace{stst\cdots}_{\text{length } i}$ and $b_i = \underbrace{tsts\cdots}_{\text{length } i}$, and note that $(s+t)^l a_i + I = (a_l + b_l)a_i + I$ and $(s+t)^l b_i + I = (a_l + b_l)b_i + I$. We easily see that as i varies in $\mathbf{N}$ these vectors run through all but finitely many spanning vectors for $F\langle s,t\rangle/I$. This shows that V/V_1 is finite dimensional; but by assumption so is V_1. It follows that V is finite dimensional, whence ϕ has trace 0. ■

5. Misleading diagonals

Recall that we may restrict our attention to a space V of countable dimension. As a finite potent φ endomorphism is the direct sum of a nilpotent one and a finite rank operator, there exists a basis of V relative to which the matrix of φ has only finitely many non-zero diagonal entries, the sum of which equals $\mathrm{Tr}_V(\varphi)$. In trying to find a counterexample to Tate's question, it would be useful to be able to compute $\mathrm{Tr}_V(\varphi)$

from its matrix on any basis, provided there are only finitely many non-zero entries in its diagonal. It turns out that this is not possible, as the following example shows.

Let F be any field and let V be a vector space over F having a basis $e_1, e_2, e_3, \ldots$ indexed by the natural numbers. Consider the nilpotent operator N on V such that

$$Ne_1 = 0,\ Ne_2 = e_1,\ Ne_3 = 0,\ Ne_4 = e_3,\ Ne_5 = 0,\ Ne_6 = e_5, \ldots$$

Consider the basis $f_1, f_2, f_3, \ldots$ of V constructed according to the following scheme:

$$e_2, e_2 + e_4, e_1 + e_4, e_4 + e_6, e_3 + e_6, e_6 + e_8, e_5 + e_8, e_8 + e_{10}, e_7 + e_{10}, e_{10} + e_{12}, e_9 + e_{12}, \ldots$$

We leave it to the reader to verify that the matrix of N relative to this basis has diagonal entries $1, 0, 0, 0, \ldots$ By adding the finite rank operator $\mathrm{diag}(-1, 0, 0, 0, \ldots)$ to N we obtain the example promised in the 'Introduction'.

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