Operator Systems: Quotients, Duals and Tensors with Applications to Connes' Embedding Conjecture

Vern Paulsen
University of Houston

Based on joint work with Douglas Farenick, Ali Kavruk, Ivan Todorov and Mark Tomforde

Regina, Saskatchewan

March, 2012

Overview

A (concrete) *operator space* is a subspace of $B(\mathcal{H})$ together with the family of induced *matrix norms.* The natural maps between operator spaces are the *completely bounded maps.*

Overview

A (concrete) *operator space* is a subspace of $B(\mathcal{H})$ together with the family of induced *matrix norms.* The natural maps between operator spaces are the *completely bounded maps.*
These have an abstract characterization(Ruan's Theorem) and "natural" morphisms, namely, the completely bounded maps.

Overview

A (concrete) *operator space* is a subspace of $B(\mathcal{H})$ together with the family of induced *matrix norms.* The natural maps between operator spaces are the *completely bounded maps.*
These have an abstract characterization(Ruan's Theorem) and "natural" morphisms, namely, the completely bounded maps.
There is a very extensive theory of operator spaces with applications to many problems in QIT.

A (concrete) *operator system* is a self-adjoint subspace of $B(\mathcal{H})$ containing the identity, together with the family of induced *matrix orders.* The natural maps between operator systems are the *completely positive maps.*

A (concrete) *operator system* is a self-adjoint subspace of $B(\mathcal{H})$ containing the identity, together with the family of induced *matrix orders.* The natural maps between operator systems are the *completely positive maps.*

These also have an even earlier abstract characterization due to Choi-Effros.

A (concrete) *operator system* is a self-adjoint subspace of $B(\mathcal{H})$ containing the identity, together with the family of induced *matrix orders.* The natural maps between operator systems are the *completely positive maps.*

These also have an even earlier abstract characterization due to Choi-Effros.

Since operator systems are the natural domains and ranges for quantum channels, many general constructions for operator systems should be useful.

A (concrete) *operator system* is a self-adjoint subspace of $B(\mathcal{H})$ containing the identity, together with the family of induced *matrix orders.* The natural maps between operator systems are the *completely positive maps.*

These also have an even earlier abstract characterization due to Choi-Effros.

Since operator systems are the natural domains and ranges for quantum channels, many general constructions for operator systems should be useful.

Today, I want to introduce you to some constructions in the operator system category and convince you of their utility.

In earlier talks here, we introduced the theory of tensor products of operator systems and using the equivalence, Connes=Kirchberg=bipartite Tsirelson, we proved that this conjecture is equivalent to some identities involving tensor products of finite dimensional, but abstract operator systems.

In earlier talks here, we introduced the theory of tensor products of operator systems and using the equivalence, Connes=Kirchberg=bipartite Tsirelson, we proved that this conjecture is equivalent to some identities involving tensor products of finite dimensional, but abstract operator systems. Today, I will recall those results but focus on duals and quotients.

In earlier talks here, we introduced the theory of tensor products of operator systems and using the equivalence, Connes=Kirchberg=bipartite Tsirelson, we proved that this conjecture is equivalent to some identities involving tensor products of finite dimensional, but abstract operator systems. Today, I will recall those results but focus on duals and quotients. Our work on duals and quotients allows us to reduce some of these questions to problems about tensor products of *concrete* operator subsystems of the finite matrices.

In earlier talks here, we introduced the theory of tensor products of operator systems and using the equivalence, Connes=Kirchberg=bipartite Tsirelson, we proved that this conjecture is equivalent to some identities involving tensor products of finite dimensional, but abstract operator systems.

Today, I will recall those results but focus on duals and quotients.

Our work on duals and quotients allows us to reduce some of these questions to problems about tensor products of *concrete* operator subsystems of the finite matrices.

Also, an open question about the tensor product of a 5 dimensional subspace of the 4×4 matrices with itself, if true, would imply the tripartite Tsirelson conjecture.

Outline

Outline

- Introduce the abstract characterization of operator systems

Outline

- Introduce the abstract characterization of operator systems
- Recall tensor product and the equivalents of Connes' conjecture

Outline

- Introduce the abstract characterization of operator systems
- Recall tensor product and the equivalents of Connes' conjecture
- Matrix ordered duals and some examples

Outline

- Introduce the abstract characterization of operator systems
- Recall tensor product and the equivalents of Connes' conjecture
- Matrix ordered duals and some examples
- Define operator system quotients and some examples

Outline

- Introduce the abstract characterization of operator systems
- Recall tensor product and the equivalents of Connes' conjecture
- Matrix ordered duals and some examples
- Define operator system quotients and some examples
- Relationship between duals and quotients

Outline

- Introduce the abstract characterization of operator systems
- Recall tensor product and the equivalents of Connes' conjecture
- Matrix ordered duals and some examples
- Define operator system quotients and some examples
- Relationship between duals and quotients
- The operator system spanned by free unitaries is just a quotient of a matrix algebra

Outline

- Introduce the abstract characterization of operator systems
- Recall tensor product and the equivalents of Connes' conjecture
- Matrix ordered duals and some examples
- Define operator system quotients and some examples
- Relationship between duals and quotients
- The operator system spanned by free unitaries is just a quotient of a matrix algebra
- Apply these facts to get concrete finite dimensional problems

Outline

- Introduce the abstract characterization of operator systems
- Recall tensor product and the equivalents of Connes' conjecture
- Matrix ordered duals and some examples
- Define operator system quotients and some examples
- Relationship between duals and quotients
- The operator system spanned by free unitaries is just a quotient of a matrix algebra
- Apply these facts to get concrete finite dimensional problems
- Introduce Kavruk's 4 dimensional quotient operator system, which gives a 16 dimensional problem that would imply tripartite Tsirelson

Abstract Characterization of Operator Systems[Choi and Effros]:

Abstract Characterization of Operator Systems[Choi and Effros]:
An (abstract) *operator system* is a $*$-vector space V, together with a collection of sets $\{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(V)_h$, and an element $e \in V$ such that

Abstract Characterization of Operator Systems[Choi and Effros]:
An (abstract) *operator system* is a $*$-vector space V, together with a collection of sets $\{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(V)_h$, and an element $e \in V$ such that

1. For each n, C_n is a spanning cone with $C_n \cap -C_n = \{0\}$.

Abstract Characterization of Operator Systems[Choi and Effros]:
An (abstract) *operator system* is a $*$-vector space V, together with a collection of sets $\{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(V)_h$, and an element $e \in V$ such that

1. For each n, C_n is a spanning cone with $C_n \cap -C_n = \{0\}$.
2. For each n and each m, whenever $X \in M_{n,m}(\mathbb{C})$ we have $X^* C_n X \subseteq C_m$.

Abstract Characterization of Operator Systems[Choi and Effros]:
An (abstract) *operator system* is a $*$-vector space V, together with a collection of sets $\{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(V)_h$, and an element $e \in V$ such that

1. For each n, C_n is a spanning cone with $C_n \cap -C_n = \{0\}$.
2. For each n and each m, whenever $X \in M_{n,m}(\mathbb{C})$ we have $X^* C_n X \subseteq C_m$.
3. The element $e_n := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix}$ is an order unit for $M_n(V)$, i.e., for each $u \in M_n(V)_h$ there exists $r > 0$ s.t. $re_n \pm u \in C_n$.

Abstract Characterization of Operator Systems[Choi and Effros]:
An (abstract) *operator system* is a $*$-vector space V, together with a collection of sets $\{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(V)_h$, and an element $e \in V$ such that

1. For each n, C_n is a spanning cone with $C_n \cap -C_n = \{0\}$.
2. For each n and each m, whenever $X \in M_{n,m}(\mathbb{C})$ we have $X^* C_n X \subseteq C_m$.
3. The element $e_n := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix}$ is an order unit for $M_n(V)$, i.e., for each $u \in M_n(V)_h$ there exists $r > 0$ s.t. $re_n \pm u \in C_n$.
4. e_n is Archimedean: For each $u \in M_n(V)$, it is the case that whenever

   $re_n + u \in C_n$ for all $r > 0$, then $u \in C_n$.

Abstract Characterization of Operator Systems[Choi and Effros]:
An (abstract) *operator system* is a $*$-vector space V, together with a collection of sets $\{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(V)_h$, and an element $e \in V$ such that

1. For each n, C_n is a spanning cone with $C_n \cap -C_n = \{0\}$.
2. For each n and each m, whenever $X \in M_{n,m}(\mathbb{C})$ we have $X^* C_n X \subseteq C_m$.
3. The element $e_n := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix}$ is an order unit for $M_n(V)$, i.e., for each $u \in M_n(V)_h$ there exists $r > 0$ s.t. $re_n \pm u \in C_n$.
4. e_n is Archimedean: For each $u \in M_n(V)$, it is the case that whenever

   $$re_n + u \in C_n \text{ for all } r > 0, \text{ then } u \in C_n.$$

Properties (1) and (2) are called a *matrix ordering*.

Abstract Characterization of Operator Systems[Choi and Effros]:
An (abstract) *operator system* is a $*$-vector space V, together with a collection of sets $\{C_n\}_{n=1}^{\infty}$, with $C_n \subseteq M_n(V)_h$, and an element $e \in V$ such that

1. For each n, C_n is a spanning cone with $C_n \cap -C_n = \{0\}$.
2. For each n and each m, whenever $X \in M_{n,m}(\mathbb{C})$ we have $X^* C_n X \subseteq C_m$.
3. The element $e_n := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix}$ is an order unit for $M_n(V)$, i.e., for each $u \in M_n(V)_h$ there exists $r > 0$ s.t. $re_n \pm u \in C_n$.
4. e_n is Archimedean: For each $u \in M_n(V)$, it is the case that whenever

   $re_n + u \in C_n$ for all $r > 0$, then $u \in C_n$.

Properties (1) and (2) are called a *matrix ordering*. The Archimedean property is a closure property for the cones.

Theorem (Choi-Effros)

If V is an abstract operator system, then there exists a Hilbert space $\mathcal{H}$ and a complete order embedding $\phi : V \to B(\mathcal{H})$ with $\phi(e) = I_{\mathcal{H}}$. In particular, $\phi(V)$ is a concrete operator subsystem of $B(\mathcal{H})$.

Theorem (Choi-Effros)

If V is an abstract operator system, then there exists a Hilbert space $\mathcal{H}$ and a complete order embedding $\phi : V \to B(\mathcal{H})$ with $\phi(e) = I_{\mathcal{H}}$. In particular, $\phi(V)$ is a concrete operator subsystem of $B(\mathcal{H})$.

Such an embedding also endows V with the structure of an operator space. In fact, the matrix norms are uniquely determined by the matrix order.

Theorem (Choi-Effros)

If V is an abstract operator system, then there exists a Hilbert space $\mathcal{H}$ and a complete order embedding $\phi : V \to B(\mathcal{H})$ with $\phi(e) = I_{\mathcal{H}}$. In particular, $\phi(V)$ is a concrete operator subsystem of $B(\mathcal{H})$.

Such an embedding also endows V with the structure of an operator space. In fact, the matrix norms are uniquely determined by the matrix order.

For $A \in M_n(V)$ we have

$$\|A\|_n = \inf \left\{ r : \begin{pmatrix} re_n & A \\ A^* & re_n \end{pmatrix} \in C_{2n} \right\}.$$

Given concrete operator systems, $\mathcal{S} \subseteq B(\mathcal{H})$ and $\mathcal{T} \subseteq B(\mathcal{K})$ the "usual" way to form a tensor product is to regard $\mathcal{S} \otimes \mathcal{T} \subseteq B(\mathcal{H} \otimes \mathcal{K})$.

Tensor Products of Operator Systems

Given concrete operator systems, $\mathcal{S} \subseteq B(\mathcal{H})$ and $\mathcal{T} \subseteq B(\mathcal{K})$ the "usual" way to form a tensor product is to regard $\mathcal{S} \otimes \mathcal{T} \subseteq B(\mathcal{H} \otimes \mathcal{K})$.
This makes $\mathcal{S} \otimes \mathcal{T}$ into a concrete operator system and this operator system is called the *spatial tensor product.*

Given concrete operator systems, $\mathcal{S} \subseteq B(\mathcal{H})$ and $\mathcal{T} \subseteq B(\mathcal{K})$ the "usual" way to form a tensor product is to regard $\mathcal{S} \otimes \mathcal{T} \subseteq B(\mathcal{H} \otimes \mathcal{K})$.
This makes $\mathcal{S} \otimes \mathcal{T}$ into a concrete operator system and this operator system is called the *spatial tensor product.*
But if we just regard $\mathcal{S}$ and $\mathcal{T}$ as abstract operator systems then, in general, there are many ways to put an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ that "respects" the operator system structures on $\mathcal{S}$ and $\mathcal{T}$.

Tensor Products of Operator Systems

Given concrete operator systems, $\mathcal{S} \subseteq B(\mathcal{H})$ and $\mathcal{T} \subseteq B(\mathcal{K})$ the "usual" way to form a tensor product is to regard $\mathcal{S} \otimes \mathcal{T} \subseteq B(\mathcal{H} \otimes \mathcal{K})$.
This makes $\mathcal{S} \otimes \mathcal{T}$ into a concrete operator system and this operator system is called the *spatial tensor product.*
But if we just regard $\mathcal{S}$ and $\mathcal{T}$ as abstract operator systems then, in general, there are many ways to put an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ that "respects" the operator system structures on $\mathcal{S}$ and $\mathcal{T}$.
In earlier work with Kavruk, Todorov and Tomforde we introduced 5 different ways to put an operator system structure on $\mathcal{S} \otimes \mathcal{T}$ and studied their properties.

In particular there is always a minimal and maximal tensor product, denoted *min*, *max*, and *min=spatial.*

In particular there is always a minimal and maximal tensor product, denoted *min, max*, and *min=spatial.* Gave characterizations of the positive cones of these tensors.

In particular there is always a minimal and maximal tensor product, denoted *min, max*, and *min=spatial.* Gave characterizations of the positive cones of these tensors.

We also identified three other important tensor products between *min* and *max.* These are:

In particular there is always a minimal and maximal tensor product, denoted *min, max*, and *min=spatial.* Gave characterizations of the positive cones of these tensors.

We also identified three other important tensor products between *min* and *max.* These are:

c the "maximal commuting"

In particular there is always a minimal and maximal tensor product, denoted *min, max*, and *min=spatial.* Gave characterizations of the positive cones of these tensors.

We also identified three other important tensor products between *min* and *max.* These are:

c the "maximal commuting"

el the "left enveloping" and *er* the "right enveloping".

In particular there is always a minimal and maximal tensor product, denoted *min, max*, and *min=spatial.* Gave characterizations of the positive cones of these tensors.

We also identified three other important tensor products between *min* and *max.* These are:

c the "maximal commuting"

el the "left enveloping" and *er* the "right enveloping".

Finally, given any pair of tensor products α and β we called an operator system $\mathcal{S}$ (α, β)-**nuclear** provided that

In particular there is always a minimal and maximal tensor product, denoted *min, max*, and *min=spatial.* Gave characterizations of the positive cones of these tensors.

We also identified three other important tensor products between *min* and *max.* These are:

c the "maximal commuting"

el the "left enveloping" and *er* the "right enveloping".

Finally, given any pair of tensor products α and β we called an operator system $\mathcal{S}$ (α, β)-**nuclear** provided that

$$\mathcal{S} \otimes_{\alpha} \mathcal{T} = \mathcal{S} \otimes_{\beta} \mathcal{T}$$

for all $\mathcal{T}$.

Let $\mathcal{S}_n = span\{1, u_1, u_1^*, ..., u_n, u_n^*\} \subseteq C^*(\mathbb{F}_n)$ denote the span of the "free" unitaries.

Let $\mathcal{S}_n = span\{1, u_1, u_1^*, ..., u_n, u_n^*\} \subseteq C^*(\mathbb{F}_n)$ denote the span of the "free" unitaries.

This operator system has the following "universal" property: Given any Hilbert space $\mathcal{H}$ and n contractions $T_1, ..., T_n \in B(\mathcal{H})$, there exists a UCP map $\phi : \mathcal{S}_n \to B(\mathcal{H})$ with $\phi(1) = I_{\mathcal{H}}$, $\phi(u_i) = T_i$.

Let $\mathcal{S}_n = span\{1, u_1, u_1^*, ..., u_n, u_n^*\} \subseteq C^*(\mathbb{F}_n)$ denote the span of the "free" unitaries.
This operator system has the following "universal" property: Given any Hilbert space $\mathcal{H}$ and n contractions $T_1, ..., T_n \in B(\mathcal{H})$, there exists a UCP map $\phi : \mathcal{S}_n \to B(\mathcal{H})$ with $\phi(1) = I_{\mathcal{H}}, \phi(u_i) = T_i$.

Theorem (KPTT2, 2010; K 2011)

The following are equivalent:

1. *Connes' embedding conjecture is true*

Let $\mathcal{S}_n = span\{1, u_1, u_1^*, ..., u_n, u_n^*\} \subseteq C^*(\mathbb{F}_n)$ denote the span of the "free" unitaries.
This operator system has the following "universal" property: Given any Hilbert space $\mathcal{H}$ and n contractions $T_1, ..., T_n \in B(\mathcal{H})$, there exists a UCP map $\phi : \mathcal{S}_n \to B(\mathcal{H})$ with $\phi(1) = I_{\mathcal{H}}$, $\phi(u_i) = T_i$.

Theorem (KPTT2, 2010; K 2011)

The following are equivalent:

1. *Connes' embedding conjecture is true*
2. $S_n \otimes_{min} S_n = S_n \otimes_c S_n$ *for all* n,

Let $\mathcal{S}_n = span\{1, u_1, u_1^*, ..., u_n, u_n^*\} \subseteq C^*(\mathbb{F}_n)$ denote the span of the "free" unitaries.
This operator system has the following "universal" property: Given any Hilbert space $\mathcal{H}$ and n contractions $T_1, ..., T_n \in B(\mathcal{H})$, there exists a UCP map $\phi : \mathcal{S}_n \to B(\mathcal{H})$ with $\phi(1) = I_{\mathcal{H}}$, $\phi(u_i) = T_i$.

Theorem (KPTT2, 2010; K 2011)

The following are equivalent:

1. *Connes' embedding conjecture is true*
2. $S_n \otimes_{min} S_n = S_n \otimes_c S_n$ *for all* n*,*
3. $S_2 \otimes_{min} S_2 = S_2 \otimes_c S_2$*(a 25 dimensional problem!),*

Let $\mathcal{S}_n = span\{1, u_1, u_1^*, ..., u_n, u_n^*\} \subseteq C^*(\mathbb{F}_n)$ denote the span of the "free" unitaries.
This operator system has the following "universal" property: Given any Hilbert space $\mathcal{H}$ and n contractions $T_1, ..., T_n \in B(\mathcal{H})$, there exists a UCP map $\phi : \mathcal{S}_n \to B(\mathcal{H})$ with $\phi(1) = I_{\mathcal{H}}$, $\phi(u_i) = T_i$.

Theorem (KPTT2, 2010; K 2011)

The following are equivalent:

1. *Connes' embedding conjecture is true*
2. $S_n \otimes_{min} S_n = S_n \otimes_c S_n$ *for all* n,
3. $S_2 \otimes_{min} S_2 = S_2 \otimes_c S_2$*(a 25 dimensional problem!),*
4. *every (min,er)-nuclear operator system is (el,c)-nuclear,*

Let $\mathcal{S}_n = span\{1, u_1, u_1^*, ..., u_n, u_n^*\} \subseteq C^*(\mathbb{F}_n)$ denote the span of the "free" unitaries.
This operator system has the following "universal" property: Given any Hilbert space $\mathcal{H}$ and n contractions $T_1, ..., T_n \in B(\mathcal{H})$, there exists a UCP map $\phi : \mathcal{S}_n \to B(\mathcal{H})$ with $\phi(1) = I_{\mathcal{H}}$, $\phi(u_i) = T_i$.

Theorem (KPTT2, 2010; K 2011)

The following are equivalent:

1. *Connes' embedding conjecture is true*
2. $S_n \otimes_{min} S_n = S_n \otimes_c S_n$ *for all* n*,*
3. $S_2 \otimes_{min} S_2 = S_2 \otimes_c S_2$*(a 25 dimensional problem!),*
4. *every (min,er)-nuclear operator system is (el,c)-nuclear,*
5. *every operator system that has the local lifting property for UCP maps has the double commutant expectation property.*

Theorem (Kavruk)

If $S_2 \otimes_{min} S_2 = S_2 \otimes_{max} S_2$, then the tripartite Tsirelson conjecture is true

Theorem (Kavruk)

If $S_2 \otimes_{min} S_2 = S_2 \otimes_{max} S_2$, *then the tripartite Tsirelson conjecture is true and*

$$C^*(F_n) \otimes_{min} C^*(\mathbb{F}_n) \otimes_{min} C^*(\mathbb{F}_n) = C^*(\mathbb{F}_n) \otimes_{max} C^*(\mathbb{F}_n) \otimes_{max} C^*(\mathbb{F}_n),$$

for all n.

Theorem (Kavruk)

If $S_2 \otimes_{min} S_2 = S_2 \otimes_{max} S_2$, then the tripartite Tsirelson conjecture is true and

$$C^*(F_n) \otimes_{min} C^*(\mathbb{F}_n) \otimes_{min} C^*(\mathbb{F}_n) = C^*(\mathbb{F}_n) \otimes_{max} C^*(\mathbb{F}_n) \otimes_{max} C^*(\mathbb{F}_n),$$

for all n.

Today: Study of duals and quotients allows us to reduce these to questions about concrete operator systems of matrices.

Given an operator system $\mathcal{S}$(or even just a matrix-ordered space), let $\mathcal{S}^d$ denote the set of all linear functionals on $\mathcal{S}$. We make $\mathcal{S}^d$ a matrix-ordered space. As follows:

Matrix Ordered Duals

Given an operator system $\mathcal{S}$(or even just a matrix-ordered space), let $\mathcal{S}^d$ denote the set of all linear functionals on $\mathcal{S}$. We make $\mathcal{S}^d$ a matrix-ordered space. As follows:
We let $\mathcal{C}_n \subseteq M_n(\mathcal{S}^d)$ be defined by $(f_{i,j}) \in \mathcal{C}_n$ iff

Matrix Ordered Duals

Given an operator system $\mathcal{S}$(or even just a matrix-ordered space), let $\mathcal{S}^d$ denote the set of all linear functionals on $\mathcal{S}$. We make $\mathcal{S}^d$ a matrix-ordered space. As follows:
We let $\mathcal{C}_n \subseteq M_n(\mathcal{S}^d)$ be defined by $(f_{i,j}) \in \mathcal{C}_n$ iff the map $\Phi : \mathcal{S} \to M_n$, given by $\Phi(x) = (f_{i,j}(x))$ is CP.

Matrix Ordered Duals

Given an operator system $\mathcal{S}$(or even just a matrix-ordered space), let $\mathcal{S}^d$ denote the set of all linear functionals on $\mathcal{S}$. We make $\mathcal{S}^d$ a matrix-ordered space. As follows:
We let $\mathcal{C}_n \subseteq M_n(\mathcal{S}^d)$ be defined by $(f_{i,j}) \in \mathcal{C}_n$ iff the map $\Phi : \mathcal{S} \to M_n$, given by $\Phi(x) = (f_{i,j}(x))$ is CP.
We often write $\mathcal{C}_n = M_n(\mathcal{S}^d)^+$.

Matrix Ordered Duals

Given an operator system $\mathcal{S}$(or even just a matrix-ordered space), let $\mathcal{S}^d$ denote the set of all linear functionals on $\mathcal{S}$. We make $\mathcal{S}^d$ a matrix-ordered space. As follows:
We let $\mathcal{C}_n \subseteq M_n(\mathcal{S}^d)$ be defined by $(f_{i,j}) \in \mathcal{C}_n$ iff the map $\Phi : \mathcal{S} \to M_n$, given by $\Phi(x) = (f_{i,j}(x))$ is CP.
We often write $\mathcal{C}_n = M_n(\mathcal{S}^d)^+$.

Theorem (PTT)

M_n is self-dual as a matrix ordered space.

Given an operator system $\mathcal{S}$(or even just a matrix-ordered space), let $\mathcal{S}^d$ denote the set of all linear functionals on $\mathcal{S}$. We make $\mathcal{S}^d$ a matrix-ordered space. As follows:
We let $\mathcal{C}_n \subseteq M_n(\mathcal{S}^d)$ be defined by $(f_{i,j}) \in \mathcal{C}_n$ iff the map $\Phi : \mathcal{S} \to M_n$, given by $\Phi(x) = (f_{i,j}(x))$ is CP.
We often write $\mathcal{C}_n = M_n(\mathcal{S}^d)^+$.

Theorem (PTT)

M_n is self-dual as a matrix ordered space. The map $\Gamma : M_n^d \to M_n$ which sends each functional to its density matrix, is a complete order isomorphism, i.e., $\Phi = (f_{i,j}) \in M_p(M_n^d)^+$ iff $\Phi : M_n \to M_p$ CP iff $(\Gamma(f_{i,j})) \in M_p(M_n)^+$.

Now for operator subsystems $\mathcal{S} \subseteq M_n$ we wish to identify $\mathcal{S}^d$, we will see that, in general, it is very different from $\mathcal{S}$.

Now for operator subsystems $\mathcal{S} \subseteq M_n$ we wish to identify $\mathcal{S}^d$, we will see that, in general, it is very different from $\mathcal{S}$.

Theorem (Choi-Effros)

If $\mathcal{S}$ is a finite dimensional operator system, then there is a positive functional $f \in \mathcal{S}^d$ such that $(\mathcal{S}^d, f)$ is an operator system.

Now for operator subsystems $\mathcal{S} \subseteq M_n$ we wish to identify $\mathcal{S}^d$, we will see that, in general, it is very different from $\mathcal{S}$.

Theorem (Choi-Effros)

If $\mathcal{S}$ is a finite dimensional operator system, then there is a positive functional $f \in \mathcal{S}^d$ such that $(\mathcal{S}^d, f)$ is an operator system.

For example, if we pick $Tr \in M_n^d$, then $\Gamma(Tr) = I_n$ and Γ defines a unit preserving COI between the operator systems (M_n, I_n) and (M_n^d, Tr).

Now for operator subsystems $\mathcal{S} \subseteq M_n$ we wish to identify $\mathcal{S}^d$, we will see that, in general, it is very different from $\mathcal{S}$.

Theorem (Choi-Effros)

If $\mathcal{S}$ is a finite dimensional operator system, then there is a positive functional $f \in \mathcal{S}^d$ such that $(\mathcal{S}^d, f)$ is an operator system.

For example, if we pick $Tr \in M_n^d$, then $\Gamma(Tr) = I_n$ and Γ defines a unit preserving COI between the operator systems (M_n, I_n) and (M_n^d, Tr).
So (M_n, I_n) and (M_n^d, Tr) really are the "same"!

Now for operator subsystems $\mathcal{S} \subseteq M_n$ we wish to identify $\mathcal{S}^d$, we will see that, in general, it is very different from $\mathcal{S}$.

Theorem (Choi-Effros)

If $\mathcal{S}$ is a finite dimensional operator system, then there is a positive functional $f \in \mathcal{S}^d$ such that $(\mathcal{S}^d, f)$ is an operator system.

For example, if we pick $Tr \in M_n^d$, then $\Gamma(Tr) = I_n$ and Γ defines a unit preserving COI between the operator systems (M_n, I_n) and (M_n^d, Tr).
So (M_n, I_n) and (M_n^d, Tr) really are the "same"!

Theorem (FP)

If $\mathcal{S}$ and $\mathcal{T}$ are finite dimensional operator systems, then $(\mathcal{S} \otimes_{min} \mathcal{T})^d = \mathcal{S}^d \otimes_{max} \mathcal{T}^d$

Now for operator subsystems $\mathcal{S} \subseteq M_n$ we wish to identify $\mathcal{S}^d$, we will see that, in general, it is very different from $\mathcal{S}$.

Theorem (Choi-Effros)

If $\mathcal{S}$ is a finite dimensional operator system, then there is a positive functional $f \in \mathcal{S}^d$ such that $(\mathcal{S}^d, f)$ is an operator system.

For example, if we pick $Tr \in M_n^d$, then $\Gamma(Tr) = I_n$ and Γ defines a unit preserving COI between the operator systems (M_n, I_n) and (M_n^d, Tr).
So (M_n, I_n) and (M_n^d, Tr) really are the "same"!

Theorem (FP)

If $\mathcal{S}$ and $\mathcal{T}$ are finite dimensional operator systems, then $(\mathcal{S} \otimes_{min} \mathcal{T})^d = \mathcal{S}^d \otimes_{max} \mathcal{T}^d$ and $(\mathcal{S} \otimes_{max} \mathcal{T})^d = \mathcal{S}^d \otimes_{min} \mathcal{T}^d$.

Operator Systems of Graphs and their Duals

Each graph G on the n vertices $V = \{1, ..., n\}$ has an edge set $E(G) \subseteq V \times V$, where we say that $(i, j) \in E(G)$ iff i and j are connected by an edge in G.

Each graph G on the n vertices $V = \{1, ..., n\}$ has an edge set $E(G) \subseteq V \times V$, where we say that $(i, j) \in E(G)$ iff i and j are connected by an edge in G.
In particular, $(i, j) \in E(G)$ iff $(j, i) \in E(G)$. For convenience we will assume that $(i, i) \in E(G)$ for all i.

Each graph G on the n vertices $V = \{1, ..., n\}$ has an edge set $E(G) \subseteq V \times V$, where we say that $(i, j) \in E(G)$ iff i and j are connected by an edge in G.
In particular, $(i, j) \in E(G)$ iff $(j, i) \in E(G)$. For convenience we will assume that $(i, i) \in E(G)$ for all i.
We now define the operator system of the graph to be

$$S_G = span\{E_{i,j} : (i, j) \in E(G)\} \subseteq M_n,$$

where $E_{i,j}$ are the canonical matrix units.

Each graph G on the n vertices $V = \{1, ..., n\}$ has an edge set $E(G) \subseteq V \times V$, where we say that $(i,j) \in E(G)$ iff i and j are connected by an edge in G.
In particular, $(i,j) \in E(G)$ iff $(j,i) \in E(G)$. For convenience we will assume that $(i,i) \in E(G)$ for all i.
We now define the operator system of the graph to be

$$S_G = span\{E_{i,j} : (i,j) \in E(G)\} \subseteq M_n,$$

where $E_{i,j}$ are the canonical matrix units.
We want to describe S_G^d, first using partially defined matrices and then using quotients.

Each functional $f : S_G \to \mathbb{C}$ we will identify with the *partially defined density matrix,*

$$\Gamma(f) = (f(E_{i,j})) : (i,j) \in E(G).$$

Each functional $f : S_G \to \mathbb{C}$ we will identify with the *partially defined density matrix,*

$$\Gamma(f) = (f(E_{i,j})) : (i,j) \in E(G).$$

Note that only some of the entries of the matrix $\Gamma(f)$ are specified, the rest can be thought of as unknowns.

Each functional $f : S_G \to \mathbb{C}$ we will identify with the *partially defined density matrix,*

$$\Gamma(f) = (f(E_{i,j})) : (i,j) \in E(G).$$

Note that only some of the entries of the matrix $\Gamma(f)$ are specified, the rest can be thought of as unknowns.
We have that f is a positive functional iff f extends to a positive functional g on M_n.

Each functional $f : S_G \to \mathbb{C}$ we will identify with the *partially defined density matrix,*

$$\Gamma(f) = (f(E_{i,j})) : (i,j) \in E(G).$$

Note that only some of the entries of the matrix $\Gamma(f)$ are specified, the rest can be thought of as unknowns.
We have that f is a positive functional iff f extends to a positive functional g on M_n. But now note that the density matrix for g corresponds to a *completion*, i.e., a specifying of the unknown entries of $\Gamma(f)$.

Each functional $f : S_G \to \mathbb{C}$ we will identify with the *partially defined density matrix,*

$$\Gamma(f) = (f(E_{i,j})) : (i,j) \in E(G).$$

Note that only some of the entries of the matrix $\Gamma(f)$ are specified, the rest can be thought of as unknowns.
We have that f is a positive functional iff f extends to a positive functional g on M_n. But now note that the density matrix for g corresponds to a *completion*, i.e., a specifying of the unknown entries of $\Gamma(f)$. Since the density matrix of g is positive, we have shown:

Each functional $f : S_G \to \mathbb{C}$ we will identify with the *partially defined density matrix,*

$$\Gamma(f) = (f(E_{i,j})) : (i,j) \in E(G).$$

Note that only some of the entries of the matrix $\Gamma(f)$ are specified, the rest can be thought of as unknowns.
We have that f is a positive functional iff f extends to a positive functional g on M_n. But now note that the density matrix for g corresponds to a *completion*, i.e., a specifying of the unknown entries of $\Gamma(f)$. Since the density matrix of g is positive, we have shown:
$f : S_G \to \mathbb{C}$ is a positive functional iff the partially defined matrix $\Gamma(f)$ can be completed to a positive matrix.

This generalizes: $(f_{r,s}) \in M_p(S_G^d)^+$ iff $(f_{r,s}) : S_G \to M_p$ is CP iff the partially defined block matrix $(\Gamma(f_{r,s})) \in M_p(M_n)$ can be completed to a positive matrix.

This generalizes: $(f_{r,s}) \in M_p(S_G^d)^+$ iff $(f_{r,s}) : S_G \to M_p$ is CP iff the partially defined block matrix $(\Gamma(f_{r,s})) \in M_p(M_n)$ can be completed to a positive matrix.
This example gives a nice introduction to the idea of *quotients.*

This generalizes: $(f_{r,s}) \in M_p(S_G^d)^+$ iff $(f_{r,s}) : S_G \to M_p$ is CP iff the partially defined block matrix $(\Gamma(f_{r,s})) \in M_p(M_n)$ can be completed to a positive matrix.

This example gives a nice introduction to the idea of *quotients.*

Note that in the Hilbert space M_n, we have
$S_G^{\perp} = span\{E_{i,j} : (i,j) \notin E(G)\}.$

This generalizes: $(f_{r,s}) \in M_p(S_G^d)^+$ iff $(f_{r,s}) : S_G \to M_p$ is CP iff the partially defined block matrix $(\Gamma(f_{r,s})) \in M_p(M_n)$ can be completed to a positive matrix.

This example gives a nice introduction to the idea of *quotients.*

Note that in the Hilbert space M_n, we have

$S_G^\perp = span\{E_{i,j} : (i,j) \notin E(G)\}$.

We can think of all completions of $\Gamma(f)$ as all matrices of the form $(f(E_{i,j})) + B$ with $B \in S_G^\perp$.

This generalizes: $(f_{r,s}) \in M_p(S_G^d)^+$ iff $(f_{r,s}) : S_G \to M_p$ is CP iff the partially defined block matrix $(\Gamma(f_{r,s})) \in M_p(M_n)$ can be completed to a positive matrix.

This example gives a nice introduction to the idea of *quotients.*

Note that in the Hilbert space M_n, we have

$S_G^{\perp} = span\{E_{i,j} : (i,j) \notin E(G)\}$.

We can think of all completions of $\Gamma(f)$ as all matrices of the form $(f(E_{i,j})) + B$ with $B \in S_G^{\perp}$.

In other words, another way to think of the partially defined matrix $\Gamma(f)$ is as a coset in the quotient vector space,

$$\Gamma(f) = (f(E_{i,j})) + S_G^{\perp} \in M_n/S_G^{\perp}.$$

This generalizes: $(f_{r,s}) \in M_p(S_G^d)^+$ iff $(f_{r,s}) : S_G \to M_p$ is CP iff the partially defined block matrix $(\Gamma(f_{r,s})) \in M_p(M_n)$ can be completed to a positive matrix.

This example gives a nice introduction to the idea of *quotients.*

Note that in the Hilbert space M_n, we have

$S_G^\perp = span\{E_{i,j} : (i,j) \notin E(G)\}$.

We can think of all completions of $\Gamma(f)$ as all matrices of the form $(f(E_{i,j})) + B$ with $B \in S_G^\perp$.

In other words, another way to think of the partially defined matrix $\Gamma(f)$ is as a coset in the quotient vector space,

$$\Gamma(f) = (f(E_{i,j})) + S_G^\perp \in M_n/S_G^\perp.$$

Thus, $(S_G^d)^+$ is exactly those cosets in $M_n/S_G^\perp$ that posess a lift to positive matrix in M_n

Quotients of Operator Systems

Given operator systems $\mathcal{S}, \mathcal{T}$ and a unital, completely positive(UCP) map, $\phi : \mathcal{S} \to \mathcal{T}$, with $\mathcal{K} = ker(\phi)$

Quotients of Operator Systems

Given operator systems $\mathcal{S}, \mathcal{T}$ and a unital, completely positive(UCP) map, $\phi : \mathcal{S} \to \mathcal{T}$, with $\mathcal{K} = \textit{ker}(\phi)$ one wishes to define a matrix order on the vector space $\mathcal{S}/\mathcal{K}$ that makes the quotient into an operator system and has the property that the induced quotient map $\dot{\phi} : \mathcal{S}/\mathcal{K} \to \mathcal{T}$ is UCP.

Quotients of Operator Systems

Given operator systems $\mathcal{S}, \mathcal{T}$ and a unital, completely positive(UCP) map, $\phi : \mathcal{S} \to \mathcal{T}$, with $\mathcal{K} = ker(\phi)$ one wishes to define a matrix order on the vector space $\mathcal{S}/\mathcal{K}$ that makes the quotient into an operator system and has the property that the induced quotient map $\dot{\phi} : \mathcal{S}/\mathcal{K} \to \mathcal{T}$ is UCP.
The relevant definitions were worked out in [KPTT2].

Quotients of Operator Systems

Given operator systems $\mathcal{S}, \mathcal{T}$ and a unital, completely positive(UCP) map, $\phi : \mathcal{S} \to \mathcal{T}$, with $\mathcal{K} = ker(\phi)$ one wishes to define a matrix order on the vector space $\mathcal{S}/\mathcal{K}$ that makes the quotient into an operator system and has the property that the induced quotient map $\dot{\phi} : \mathcal{S}/\mathcal{K} \to \mathcal{T}$ is UCP.
The relevant definitions were worked out in [KPTT2].

Definition

A subspace $\mathcal{K} \subset \mathcal{S}$ is a *kernel* if there are an operator system $\mathcal{T}$ and a completely positive linear map $\phi : \mathcal{S} \to \mathcal{T}$ such that $\mathcal{K} = \ker \phi$.

Given operator systems $\mathcal{S}, \mathcal{T}$ and a unital, completely positive(UCP) map, $\phi : \mathcal{S} \to \mathcal{T}$, with $\mathcal{K} = \mathit{ker}(\phi)$ one wishes to define a matrix order on the vector space $\mathcal{S}/\mathcal{K}$ that makes the quotient into an operator system and has the property that the induced quotient map $\dot{\phi} : \mathcal{S}/\mathcal{K} \to \mathcal{T}$ is UCP.
The relevant definitions were worked out in [KPTT2].

Definition

A subspace $\mathcal{K} \subset \mathcal{S}$ is a *kernel* if there are an operator system $\mathcal{T}$ and a completely positive linear map $\phi : \mathcal{S} \to \mathcal{T}$ such that $\mathcal{K} = \ker \phi$.

If $\mathcal{K} \subset \mathcal{S}$ is a kernel, then define $C_n(\mathcal{S}/\mathcal{K}) \subset M_n(\mathcal{S}/\mathcal{K})_{\mathrm{sa}}$ to be

$$\{\dot{H} \,:\, \forall \varepsilon > 0 \, \exists \, K_\varepsilon \in M_n(\mathcal{K})_{\mathrm{sa}} \text{ such that } \varepsilon 1 + H + K_\varepsilon \in M_n(\mathcal{S})_+\} \,.$$

The collection $\{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}$ is a family of cones that endow $\mathcal{S}/\mathcal{K}$ with the structure of an operator system with (Archimedean) order unit $\dot{1} = q(1)$.

The collection $\{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}$ is a family of cones that endow $\mathcal{S}/\mathcal{K}$ with the structure of an operator system with (Archimedean) order unit $\dot{1} = q(1)$.

Definition

The operator system $(\mathcal{S}/\mathcal{K}, \{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}, q(1))$ *is called the* **quotient operator system**.

The collection $\{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}$ is a family of cones that endow $\mathcal{S}/\mathcal{K}$ with the structure of an operator system with (Archimedean) order unit $\dot{1} = q(1)$.

Definition

The operator system $(\mathcal{S}/\mathcal{K}, \{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}, q(1))$ *is called the* **quotient operator system**.

Definition

Given operator systems $\mathcal{S}, \mathcal{T}$ *and a UCP map* $\phi : \mathcal{S} \to \mathcal{T}$ *with* $\mathcal{K} = ker(\phi)$, *we call* ϕ *a* **complete quotient map** *if the induced map* $\dot{\phi} : \mathcal{S}/\mathcal{K} \to \mathcal{T}$ *is a complete order isomorphism, i.e., UCP, onto and* $\dot{\phi}^{-1}$ *is UCP.*

The collection $\{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}$ is a family of cones that endow $\mathcal{S}/\mathcal{K}$ with the structure of an operator system with (Archimedean) order unit $\dot{1} = q(1)$.

Definition

The operator system $(\mathcal{S}/\mathcal{K}, \{C_n(\mathcal{S}/\mathcal{K})\}_{n\in\mathbb{N}}, q(1))$ is called the **quotient operator system**.

Definition

Given operator systems $\mathcal{S}, \mathcal{T}$ and a UCP map $\phi : \mathcal{S} \to \mathcal{T}$ with $\mathcal{K} = ker(\phi)$, we call ϕ a **complete quotient map** *if the induced map $\dot{\phi} : \mathcal{S}/\mathcal{K} \to \mathcal{T}$ is a complete order isomorphism, i.e., UCP, onto and $\dot{\phi}^{-1}$ is UCP.*

Note: Complete order isomorphisms(COI) are the "natural" identifications in the category whose objects are operator systems and whose morphisms are the UCP maps.

Let $S \subseteq M_n$, write $\iota : S \to M_n$ for the inclusion map and let $\iota^d : M_n^d \to S^d$ be the Banach space adjoint.

Let $S \subseteq M_n$, write $\iota : S \to M_n$ for the inclusion map and let $\iota^d : M_n^d \to S^d$ be the Banach space adjoint.

Theorem (FP)

ι^d is a complete quotient map. Consequently, S^d is COI to the quotient operator system $M_n/ker(\iota^d) = M_n/\overline{S}^{\perp}$.

Let $S \subseteq M_n$, write $\iota : S \to M_n$ for the inclusion map and let $\iota^d : M_n^d \to S^d$ be the Banach space adjoint.

Theorem (FP)

ι^d is a complete quotient map. Consequently, S^d is COI to the quotient operator system $M_n/ker(\iota^d) = M_n/\overline{S}^{\perp}$.

Corollary (FP)

S_G^d is COI to $M_n/S_G^{\perp}$.

Free Unitaries are a Quotient

Let $\mathcal{T}_{n+1} \subseteq M_{n+1}$ denote the set of tridiagonal matrices, and let $\mathcal{J}_{n+1} \subseteq M_{n+1}$ denote the set of diagonal matrices of trace 0.

Free Unitaries are a Quotient

Let $\mathcal{T}_{n+1} \subseteq M_{n+1}$ denote the set of tridiagonal matrices, and let $\mathcal{J}_{n+1} \subseteq M_{n+1}$ denote the set of diagonal matrices of trace 0.

Theorem (FP)

The UCP map $\psi : \mathcal{T}_{n+1} \to \mathcal{S}_n$, *given by* $\psi(E_{i,i}) = \frac{1}{n+1}$, $\psi(E_{i,i+1}) = \frac{u_i}{n}$ *and* $\psi(E_{i+1,i}) = \frac{u_i^*}{n}$ *is a complete quotient map. Hence,* $\mathcal{T}_{n+1}/\mathcal{J}_{n+1} = \mathcal{S}_n$ *up to COI.*

Free Unitaries are a Quotient

Let $\mathcal{T}_{n+1} \subseteq M_{n+1}$ denote the set of tridiagonal matrices, and let $\mathcal{J}_{n+1} \subseteq M_{n+1}$ denote the set of diagonal matrices of trace 0.

Theorem (FP)

The UCP map $\psi : \mathcal{T}_{n+1} \to \mathcal{S}_n$, given by $\psi(E_{i,i}) = \frac{1}{n+1}$, $\psi(E_{i,i+1}) = \frac{u_i}{n}$ and $\psi(E_{i+1,i}) = \frac{u_i^*}{n}$ is a complete quotient map. Hence, $\mathcal{T}_{n+1}/\mathcal{J}_{n+1} = \mathcal{S}_n$ up to COI.

Corollary (FP)

Let $A_0, ..., A_n \in B(\mathcal{H})$. *Then the following are equivalent:*

- $P = A_0 \otimes 1 + (\sum_{i=1}^{n} A_i \otimes u_i) + (\sum_{i=1}^{n} A_i \otimes u_i)^*$ *is positive in* $C^*(\mathbb{F}_n) \otimes_{min} B(\mathcal{H})$,

Corollary (FP)

Let $A_0, ..., A_n \in B(\mathcal{H})$. Then the following are equivalent:

- $P = A_0 \otimes 1 + (\sum_{i=1}^{n} A_i \otimes u_i) + (\sum_{i=1}^{n} A_i \otimes u_i)^*$ *is positive in* $C^*(\mathbb{F}_n) \otimes_{min} B(\mathcal{H})$,
- *there exist* $B_1, ..., B_{n+1} \in B(\mathcal{H})$ *with* $A_0 = \sum_{i=1}^{n+1} B_i$ *such that the tri-diagonal operator matrix* $\sum_{i=1}^{n+1} B_i \otimes E_{i,i} + (\sum_{i=1}^{n} A_i \otimes E_{i,i+1}) + (\sum_{i=1}^{n} A_i \otimes E_{i+1,i})^*$ *is positive in* $B(\mathcal{H}) \otimes M_{n+1} = B(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$,

Corollary (FP)

Let $A_0, ..., A_n \in B(\mathcal{H})$. Then the following are equivalent:

- $P = A_0 \otimes 1 + (\sum_{i=1}^n A_i \otimes u_i) + (\sum_{i=1}^n A_i \otimes u_i)^*$ *is positive in* $C^*(\mathbb{F}_n) \otimes_{min} B(\mathcal{H})$,
- *there exist* $B_1, ..., B_{n+1} \in B(\mathcal{H})$ *with* $A_0 = \sum_{i=1}^{n+1} B_i$ *such that the tri-diagonal operator matrix* $\sum_{i=1}^{n+1} B_i \otimes E_{i,i} + (\sum_{i=1}^n A_i \otimes E_{i,i+1}) + (\sum_{i=1}^n A_i \otimes E_{i+1,i})^*$ *is positive in* $B(\mathcal{H}) \otimes M_{n+1} = B(\mathcal{H} \oplus \cdots \oplus \mathcal{H})$,
- *there exist* $C_1, D_1, ..., C_n, D_n \in B(\mathcal{H})$ *such that* $P = \sum_{i=1}^n (C_i \otimes 1 + D_i \otimes u_i)^*(C_i \otimes 1 + D_i \otimes u_i)$.

$$\mathcal{V}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{2,2}^k = a_{1,1}^{k+1}\} \subseteq M_{2(n-1)}$$

$$\mathcal{V}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{2,2}^k = a_{1,1}^{k+1}\} \subseteq M_{2(n-1)}$$

$$\mathcal{U}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{i,i}^k = a_{j,j}^l, \forall i, j, k, l\} \subseteq \mathcal{V}_n$$

$$\mathcal{V}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{2,2}^k = a_{1,1}^{k+1}\} \subseteq M_{2(n-1)}$$

$$\mathcal{U}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{i,i}^k = a_{j,j}^l, \forall i, j, k, l\} \subseteq \mathcal{V}_n$$

Theorem (FKP)

$\mathcal{T}_n^d = \mathcal{V}_n$ and $\mathcal{S}_n^d = \mathcal{U}_{n+1}$.

$$\mathcal{V}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{2,2}^k = a_{1,1}^{k+1}\} \subseteq M_{2(n-1)}$$

$$\mathcal{U}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{i,i}^k = a_{j,j}^l, \forall i,j,k,l\} \subseteq \mathcal{V}_n$$

Theorem (FKP)

$\mathcal{T}_n^d = \mathcal{V}_n$ and $\mathcal{S}_n^d = \mathcal{U}_{n+1}$. Consequently,

- *if $\mathcal{U}_3 \otimes_{min} \mathcal{U}_3 = \mathcal{U}_3 \otimes_{max} \mathcal{U}_3$, then the tripartite Tsirelson conjecture is true,*

$$\mathcal{V}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{2,2}^k = a_{1,1}^{k+1}\} \subseteq M_{2(n-1)}$$

$$\mathcal{U}_n = \{\bigoplus_{k=1}^{n-1} \begin{pmatrix} a_{1,1}^k & a_{1,2}^k \\ a_{2,1}^k & a_{2,2}^k \end{pmatrix} : a_{i,i}^k = a_{j,j}^l, \forall i,j,k,l\} \subseteq \mathcal{V}_n$$

Theorem (FKP)

$\mathcal{T}_n^d = \mathcal{V}_n$ *and* $\mathcal{S}_n^d = \mathcal{U}_{n+1}$. *Consequently,*

- *if* $\mathcal{U}_3 \otimes_{min} \mathcal{U}_3 = \mathcal{U}_3 \otimes_{max} \mathcal{U}_3$, *then the tripartite Tsirelson conjecture is true,*
- $\mathcal{U}_3 \otimes_{min} \mathcal{U}_3 = \mathcal{U}_3 \otimes_{c^d} \mathcal{U}_3$ *iff Connes' embedding conjecture is true.*

Kavruk's Operator System

Kavruk's Operator System

Let $\mathcal{J} = span\{(1,1,1,-1,-1)\} \subseteq \ell^{\infty}_5 = \mathcal{D}_5$,

Kavruk's Operator System

Let $\mathcal{J} = span\{(1,1,1,-1,-1)\} \subseteq \ell_5^\infty = \mathcal{D}_5$, so that $\mathcal{K}_4 = \mathcal{D}_5/\mathcal{J}$ is a 4-dimensional operator system.

Kavruk's Operator System

Let $\mathcal{J} = span\{(1, 1, 1, -1, -1)\} \subseteq \ell_5^\infty = \mathcal{D}_5$, so that $\mathcal{K}_4 = \mathcal{D}_5/\mathcal{J}$ is a 4-dimensional operator system.

Theorem (Kavruk)

Connes' embedding conjecture is true iff $\mathcal{K}_4 \otimes_{min} \mathcal{K}_4 = \mathcal{K}_4 \otimes_c \mathcal{K}_4$(a 16 dimensional problem!). If $\mathcal{K}_4 \otimes_{min} \mathcal{K}_4 = \mathcal{K}_4 \otimes_{max} \mathcal{K}_4$, then the tripartite Tsirelson is true.

Let $\mathcal{J} = span\{(1,1,1,-1,-1)\} \subseteq \ell^\infty_5 = \mathcal{D}_5$, so that $\mathcal{K}_4 = \mathcal{D}_5/\mathcal{J}$ is a 4-dimensional operator system.

Theorem (Kavruk)

Connes' embedding conjecture is true iff $\mathcal{K}_4 \otimes_{min} \mathcal{K}_4 = \mathcal{K}_4 \otimes_c \mathcal{K}_4$ *(a 16 dimensional problem!). If* $\mathcal{K}_4 \otimes_{min} \mathcal{K}_4 = \mathcal{K}_4 \otimes_{max} \mathcal{K}_4$, *then the tripartite Tsirelson is true.*

Note that the operator system $\mathcal{D}_5$ is also self-dual!

Kavruk's Operator System

Let $\mathcal{J} = span\{(1,1,1,-1,-1)\} \subseteq \ell^\infty_5 = \mathcal{D}_5$, so that $\mathcal{K}_4 = \mathcal{D}_5/\mathcal{J}$ is a 4-dimensional operator system.

Theorem (Kavruk)

Connes' embedding conjecture is true iff $\mathcal{K}_4 \otimes_{min} \mathcal{K}_4 = \mathcal{K}_4 \otimes_c \mathcal{K}_4$(a 16 dimensional problem!). If $\mathcal{K}_4 \otimes_{min} \mathcal{K}_4 = \mathcal{K}_4 \otimes_{max} \mathcal{K}_4$, then the tripartite Tsirelson is true.

Note that the operator system $\mathcal{D}_5$ is also self-dual!
It follows from the results of FP, that
$\mathcal{K}_4^d = \{(x_1, ..., x_5) \in \mathcal{D}_5 : x_1 + x_2 + x_3 - x_4 - x_5 = 0\}$.

Kavruk's Operator System

Let $\mathcal{J} = span\{(1,1,1,-1,-1)\} \subseteq \ell_5^\infty = \mathcal{D}_5$, so that $\mathcal{K}_4 = \mathcal{D}_5/\mathcal{J}$ is a 4-dimensional operator system.

Theorem (Kavruk)

Connes' embedding conjecture is true iff $\mathcal{K}_4 \otimes_{min} \mathcal{K}_4 = \mathcal{K}_4 \otimes_c \mathcal{K}_4$(a 16 dimensional problem!). If $\mathcal{K}_4 \otimes_{min} \mathcal{K}_4 = \mathcal{K}_4 \otimes_{max} \mathcal{K}_4$, then the tripartite Tsirelson is true.

Note that the operator system $\mathcal{D}_5$ is also self-dual!
It follows from the results of FP, that
$\mathcal{K}_4^d = \{(x_1, ..., x_5) \in \mathcal{D}_5 : x_1 + x_2 + x_3 - x_4 - x_5 = 0\}$.

Corollary

Connes' embedding conjecture is true iff $\mathcal{K}_4^d \otimes_{min} \mathcal{K}_4^d = \mathcal{K}_4^d \otimes_{c^d} \mathcal{K}_4^d$. If $\mathcal{K}_4^d \otimes_{min} \mathcal{K}_4^d = \mathcal{K}_4^d \otimes_{max} \mathcal{K}_4^d$, then the tripartite Tsirelson is true.

To see what this means, concretely...at ground level must decide if the following cones are equal:

To see what this means, concretely...at ground level must decide if the following cones are equal:

$(\mathcal{K}_4 \otimes_{min} \mathcal{K}_4)^+$

To see what this means, concretely...at ground level must decide if the following cones are equal:

$(\mathcal{K}_4 \otimes_{min} \mathcal{K}_4)^+ = \{(x_{i,j}) \in M_5 :$

To see what this means, concretely...at ground level must decide if the following cones are equal:

$$(\mathcal{K}_4 \otimes_{min} \mathcal{K}_4)^+ = \{(x_{i,j}) \in M_5 :$$
$$x_{i,j} \geq 0, (1,1,1,-1,-1)(x_{i,j}) = \vec{0}, (x_{i,j})(1,1,1,-1,-1)^t = \vec{0}\}$$

To see what this means, concretely...at ground level must decide if the following cones are equal:

$(\mathcal{K}_4 \otimes_{min} \mathcal{K}_4)^+ = \{(x_{i,j}) \in M_5 :$
$x_{i,j} \geq 0, (1,1,1,-1,-1)(x_{i,j}) = \vec{0}, (x_{i,j})(1,1,1,-1,-1)^t = \vec{0}\}$

$(\mathcal{K}_4 \otimes_{max} \mathcal{K}_4)^+$

To see what this means, concretely...at ground level must decide if the following cones are equal:

$$(\mathcal{K}_4 \otimes_{min} \mathcal{K}_4)^+ = \{(x_{i,j}) \in M_5 : x_{i,j} \geq 0, (1,1,1,-1,-1)(x_{i,j}) = \vec{0}, (x_{i,j})(1,1,1,-1,-1)^t = \vec{0}\}$$

$$(\mathcal{K}_4 \otimes_{max} \mathcal{K}_4)^+ = \{(Tr(X_i Y_j)) \in M_5 : \exists n, X_i, Y_j \in M_n^+,$$

To see what this means, concretely...at ground level must decide if the following cones are equal:

$$(\mathcal{K}_4 \otimes_{min} \mathcal{K}_4)^+ = \{(x_{i,j}) \in M_5 : \\ x_{i,j} \geq 0, (1,1,1,-1,-1)(x_{i,j}) = \vec{0}, (x_{i,j})(1,1,1,-1,-1)^t = \vec{0}\}$$

$$(\mathcal{K}_4 \otimes_{max} \mathcal{K}_4)^+ = \{(Tr(X_i Y_j)) \in M_5 : \exists n, X_i, Y_j \in M_n^+, \\ X_1 + X_2 + X_3 - X_4 - X_5 = 0, Y_1 + Y_2 + Y_3 + -Y_4 - Y_5 = 0\}$$