Infinite Dimensional Versions of the Schur-Horn Theorem

M Argerami

University of Regina

SK Analysis Day II
Saskatoon, March 17, 2012
The Pythagorean Theorem

Pythagorean Theorem (PT):

If $x \gamma h y$, $\gamma = \pi/2$, then $x^2 + y^2 = h^2$.

A less known fact, yet interesting and simple, is that the converse holds:

Carpenter’s Theorem (CT):

If $x \gamma h y$, $x^2 + y^2 = h^2$, then $\gamma = \pi/2$. 
Consider $\mathbb{R}^2$ with the usual Euclidean structure, and canonical basis \{e_1, e_2\}. In this context,

**PT2**

If $x = c_1 e_1 + c_2 e_2$ and $\|x\| = 1$, then $|c_1|^2 + |c_2|^2 = 1$. 

(that is, Pythagoras=Parseval)

**CT2**

If $t_1, t_2 \in \mathbb{R}^+$, $t_1^2 + t_2^2 = 1$, then there exists $x \in \mathbb{R}^2$ with $\|x\| = 1$ and $\|P_{Re_1}x\| = t_1$, $\|P_{Re_2}x\| = t_2$. 
Since both $x$ and $e_1$ have norm equal to 1,
$$\| P_{Re_1} x \|^2 = |\langle x, e_1 \rangle|^2 = \| P_{Rx} e_1 \|^2.$$ Thus

**PT3**

If $K \subset \mathbb{R}^2$ is a one-dimensional subspace, then
$$\| P_K e_1 \|^2 + \| P_K e_2 \|^2 = 1.$$

**CT3**

If $t_1, t_2 \in \mathbb{R}^+$, $t_1^2 + t_2^2 = 1$, then there exists $K \subset \mathbb{R}^2$, one-dimensional, with $\| P_K e_1 \| = t_1$, $\| P_K e_2 \| = t_2$. 
3. PT and CT in $\mathbb{R}^n$

If we consider $\mathbb{R}^n$ with canonical basis $\{e_j\}_{j=1}^n$,

**PT4**

If $K \subset \mathbb{R}^n$ is a one-dimensional subspace, then $\sum_{j=1}^n \|P_K e_j\|^2 = 1$.

**CT4**

If $t_1, \ldots, t_n \in [0, 1]$, $\sum_{j=1}^n t_j^2 = 1$, then there exists $K \subset \mathbb{R}^n$, one-dimensional, with $\|P_K e_j\| = t_j$, $j = 1, \ldots, n$. 

Argerami (U of R)  
Schur-Horn Theorem  
SK Analysis Day II
A natural generalization would be to allow $K$ to have different dimensions:

**PT5**

If $K \subset \mathbb{R}^n$ is an $m$-dimensional subspace, then $\sum_{j=1}^{n} \|P_k e_j\|^2 = m$.

**CT5**

If $\{t_j\}_{j=1}^{n} \subset [0, 1]$, $\sum_{j=1}^{n} t_j^2 = m$, then there exists $K \subset \mathbb{R}^n$, $m$-dimensional, with $\|P_k e_j\| = t_j$, $j = 1, \ldots, n$.

Do these two hold?
Reformulation of PT5 and CT5

Since
\[ \| P_K e_j \|^2 = \langle P_K e_j, P_K e_j \rangle = \langle P_K e_j, e_j \rangle, \]
we have
\[ \sum_{j=1}^{n} \| P_K e_j \|^2 = \sum_{j=1}^{n} \langle P_K e_j, e_j \rangle = \text{Tr} (P_K). \]

So PT5 and CT5 reformulate as

**PT6**
If \( K \subset \mathbb{R}^n \) is an \( m \)-dimensional subspace, then \( \text{Tr} (P_K) = m. \)

**CT6**
If \( t_1, \ldots, t_n \in [0, 1], \sum_{j=1}^{n} t_j^2 = m \), then there exists \( K \subset \mathbb{R}^n \) such that the diagonal of \( P_K \) is \( (t_1^2, \ldots, t_n^2) \).
Up to PT4/CT4, all the results are trivial. So is PT5/6. What happens with CT5/CT6?

Given $t_1, \ldots, t_n$ with $t_1^2 + \cdots + t_n^2 = m \in \mathbb{N}$, we look for $K \subset \mathbb{R}^n$ with $\|P_K e_j\| = t_j$ for all $j$. That is, we look for a matrix $P_K$ such that $P_K^2 = P_K^T = P_K$, with a pre-determined diagonal: $(P_K)_{jj} = t_j^2$.

Is that possible? It is not obvious: the conditions induce $n(n+1)/2$ equations in $n(n-1)/2$ unknowns.

In $2 \times 2$: $P_K = \begin{bmatrix} t^2 & x \\ x & 1 - t^2 \end{bmatrix}$

\[
\begin{align*}
t^2 &= t^4 + x^2 \\
1 - t^2 &= (1 - t^2)^2 + x^2
\end{align*}
\]

$x = \sqrt{t^2 - t^4}$, and the two equations are compatible. Is it always the case?
4. Majorization

Given $x, y \in \mathbb{R}^n$, we say that $x$ is majorized by $y$ ($x \prec y$) if

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j, \quad k = 1, \ldots, n - 1; \quad \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$ 

For instance, $(\frac{1}{n}, \ldots, \frac{1}{n}) \prec (x_1, \ldots, x_n) \prec (1, 0, \ldots, 0)$ for all $(x_1, \ldots, x_n) \in [0, 1]^n$ with $x_1 + \cdots + x_n = 1$.

As a consequence of Birkhoff’s Theorem,

$$\{ x \in \mathbb{R}^n : x \prec y \} = \text{conv}\{ \mathbb{S}_n y \}.$$
5. The Schur-Horn Theorem

For $A \in M_n(\mathbb{R})$, denote $\text{diag}(A) = (a_{11}, a_{22}, \ldots, a_{nn}) \in \mathbb{R}^n$, $\lambda(A) =$eigenvalues of $A$ counted with multiplicity.

**Theorem (Schur, 1923)**

Let $A \in M_n(\mathbb{R})$ be symmetric. Then $\text{diag}(A) \prec \lambda(A)$.

**Theorem (A. Horn, 1956)**

Let $x, y \in \mathbb{R}^n$ with $x \prec y$. Then there exists $A \in M_n(\mathbb{R})$, symmetric, with $\text{diag}(A) = x$, $\lambda(A) = y$.

**Theorem (Schur-Horn)**

$$\{M_x \in \mathbb{R}^n : x \prec y\} = D(\{u M_y u^* : u \in O_n\})$$.
6. Proof of CT6

CT6

if \( t_1, \ldots, t_n \in [0, 1] \), \( \sum_{j=1}^{n} t_j^2 = m \), then there exists \( K \subset \mathbb{R}^n \) such that the diagonal of \( P_K \) is \( (t_1^2, \ldots, t_n^2) \).

We need to construct a projection \( P \) with pre-determined diagonal \((t_1^2, \ldots, t_n^2)\).

The key observation:

\((t_1^2, \ldots, t_n^2) \prec (1, \ldots, 1, 0, \ldots, 0)\). By Schur-Horn, there exists \( U \), unitary, such that \((t_1^2, \ldots, t_n^2)\) is the diagonal of

\[
U \cdot \begin{bmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & 0 \\
& & & 0 & \\
& & & & \ddots \\
0 \\
& & & & & \\
& & & & & & \\
& & & & & & & \\
& & & & & & & & &
\end{bmatrix} \cdot U^*.
\]
We change $\mathbb{R}^n$ for an infinite-dimensional Hilbert space $H$.

**Theorem (Kadison)**

Given $\{e_j\}_{j \in J}$ o.n.b. of $H$ and $\{t_j\}_{j \in J} \subset [0, 1]$, TFSAE:

1. There exists $K \subset H$, $m$-dimensional, such that the diagonal of $P_K$ is $\{t_j^2\}_{j \in J}$;
2. $\sum_{j \in J} t_j^2 = m$.

What about the case $\sum t_j^2 = \infty$? If $E$ is a projection with $\{t_j^2\}$ in the diagonal, then so is $I - E$, and its diagonal is $\{1 - t_j^2\}$; if $\text{Tr}(I - E) < \infty$, it has to be an integer: $\sum_j (1 - t_j^2) \in \mathbb{N}$.

For instance, there is no projection in $B(H)$ with $\{1 - 1/j^2\}$ in the diagonal.
Theorem (Kadison)
Given \( \{ e_j \}_{j \in J} \) an o.n.b. of \( H \) and \( \{ t_j \}_{j \in J} \subset [0, 1] \), TFSAE:

1. There exists \( K \subset H \) subspace with co-dimension \( m \) such that the diagonal of \( P_K \) is \( \{ t_j^2 \}_{j \in J} \);

2. \( \sum_{j \in J} (1 - t_j^2) = m \).

What happens when both sums are infinite?

Theorem (Kadison)
Given \( \{ e_j \}_{j \in J} \) a o.n.b. of \( H \) and \( \{ t_j \}_{j \in J} \subset [0, 1] \), TFSAE:

1. There exists \( K \subset H \) with infinite dimension and co-dimension, such that the diagonal of \( P_K \) is \( \{ t_j^2 \}_{j \in J} \);

2. \( \sum_{j \in J} t_j^2 = \infty \) and \( \sum_{j \in J} (1 - t_j^2) = \infty \); and either (i) or (ii) holds:

   (i) \( \sum_{t_j^2 \leq 1/2} t_j^2 = \infty \) or \( \sum_{t_j^2 > 1/2} (1 - t_j^2) = \infty \);

   (ii) \( \sum_{t_j^2 \leq 1/2} t_j^2 < \infty \), \( \sum_{t_j^2 > 1/2} (1 - t_j^2) < \infty \), and

   \( \sum_{t_j^2 \leq 1/2} t_j^2 - \sum_{t_j^2 > 1/2} (1 - t_j^2) \in \mathbb{Z} \).
Schur-Horn in $B(H)$: majorization

How do we phrase Schur-Horn in $B(H)$? Recall that SH is

$$\{M_x \in \mathbb{R}^n : x \prec y\} = D(\{u M_y u^* : u \in O_n\}).$$

What’s the meaning of $x \prec y$ for $x, y \in \ell^\infty (\mathbb{N}) \subset B(H)$? If we take the obvious generalization, we would have

$$\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow, \quad k \in \mathbb{N}; \quad \sum_{j=1}^\infty x_j = \sum_{k=1}^\infty y_j.$$

Any problems with this? **Yes.** In finite dimension,

$$x \prec y, \; y \prec x \implies x \in S y.$$ 

Consider $x = (1, 1, 1, \ldots), \; y = (1, 0, 1, 0, \ldots)$; then $x \prec y, \; y \prec x$, but they are really far from being permutations of each other!
The right definition of majorization in $\ell_\infty^\mathbb{R}(\mathbb{N}) \subset B(H)$

Neumann’s idea:
In finite dimension, if $\sum_{j=1}^n x_j = \sum_{j=1}^n y_j$, then

$$\sum_{j=1}^k x_j \uparrow \leq \sum_{j=1}^k y_j \uparrow, \quad k = 1, \ldots, n \iff \sum_{j=1}^k x_j \uparrow \geq \sum_{j=1}^k y_j \uparrow, \quad k = 1, \ldots, n$$

For $x \in \ell_\infty^\mathbb{R}(\mathbb{N})$, $k \in \mathbb{N}$, let

$$U_k(x) = \sup\{\sum_{j \in F} x_j : |F| = k\}, \quad L_k(x) = \inf\{\sum_{j \in F} x_j : |f| = k\}.$$  

We define

$$x \prec y \iff U_k(x) \leq U_k(y), \quad L_k(x) \geq L_k(y), \quad k \in \mathbb{N}.$$  

So

$$(1, 1, 1, \ldots) \prec (1, 0, 1, 0, \ldots), \quad \text{but} \quad (1, 0, 1, 0, \ldots) \not\prec (1, 1, 1, 1, \ldots)$$
Schur-Horn in $B(H)$: continued

An algebraic generalization of an orthonormal basis is a masa (maximal abelian sub-algebra).

Let $A \subset B(H)$ be an atomic masa. Then there exists a conditional expectation $E : B(H) \to A$ (that is a projection of norm 1). So SH would be

$$\{M_a \in A : a \prec b\} = E(\{uM_bu^* : u \in \mathcal{U}(H)\})$$

But it is false: a norm-closure is required on the right-hand-side:

**Theorem (A. Neumann, 1999)**

*For any $y \in \ell_\infty^\infty(\mathbb{N}),$

$$\{x : x \prec y\} = \overline{E_A(\{uM_yu^* : u \in \mathcal{U}(H)\})}$$

Is the closure really necessary? **Yes.**
Necessity of the closure

\[ x = (1, 1, 1, \ldots) \prec y = (1, 0, 1, 0, \ldots), \]

so \( I = M_x \in \overline{E(\{uM_yu^* : u \in \mathcal{U}(H)\})} \), but

\[ I \not\in E(\{uM_yu^* : u \in \mathcal{U}(H)\}) \]
8. II\(_1\)-factors

A II\(_1\)-factor is a \(*\)-subalgebra \(M \subset B(H)\), closed with the pointwise topology (i.e., \(M\) is a von Neumann algebra), with trivial centre, infinite dimensional, and with a \textit{finite trace} \(\tau\):

1. \(\tau : M \to \mathbb{C}\) is \(*\)-linear;
2. \(\tau\) is normal (i.e., strongly continuous);
3. \(\tau\) is faithful (i.e. \(\tau(a^*a) = 0 \implies a = 0\));
4. \(\tau(ab) = \tau(ba)\) for all \(a, b \in M\).

II\(_1\) factors can often be thought of as a continuous generalization of full-matrix algebras. Although not obvious, the conditions above imply that for every \(t \in [0, 1]\), there exists a projection \(p \in M\) with \(\tau(p) = t\).

Like in \(B(H)\), in lieu of an o.n.b. we consider a masa \(A \subset M\). It is well-known that there always exists a projection \(E : M \to A\) of norm one (i.e. a \textit{conditional expectation}).
Majorization in $\text{II}_1$-factors.

Consider $a \in M^{sa}$.
The spectral scale:

$$\lambda_t(a) = \min\{s : \tau(p^a(s, \infty)) \leq t\}.$$

We say that $a \prec b$ if

$$\int_0^s \lambda_t(a) \, dt \leq \int_0^s \lambda_t(b) \, dt, \quad s \in [0, 1]$$

with equality in $s = 1$. Note that

$$\int_0^1 \lambda_t(a) \, dt = \tau(a).$$
So we can ask about what Schur-Horn would look like in this setting:

\[ E(\mathcal{U}_M(b)) \overset{?}{=} \{ a \in A : a \prec b \} \]

(\text{notation: } \mathcal{U}_M(b) = \{ ubu^* : u \in \mathcal{U}_M \})

Two points of view: characterize majorization/diagonals of selfadjoint operators.

**Theorem (Argerami-Massey 2007)**

*In the situation above, we have the equality*

\[ \overline{E(\mathcal{U}_M(b))}^{sot} = \{ a \in A : a \prec b \} \]
As in the finite-dimensional case, the *Carpenter’s Theorem* is a special case of Schur-Horn: given a projection $p \in M$, does

$$E(\{ upu^* : u \in \mathcal{U}_M \}) = \{ a \in A_1^+ : a \prec p \}$$

hold?

Note that

$$\{ a \in A : a \prec p \} = \{ a \in A : a \prec p \} = \{ a \in A_1^+ : a \prec p \}$$

When $M = B(H)$ (not a $\text{II}_1$-factor!), Kadison’s obstruction shows that the equality fails.

In $\text{II}_1$-factors, the question is still open. Even the easier question

$$\exists p \in M : E(p) = \frac{1}{\sqrt{2}} ?$$

has only been answered a few months ago:
Theorem (Bhat-Ravichandran, 2011)

Let $A$ be a masa in the $II_1$-factor $M$. If $x \in M^{sa}$ has finite spectrum, $a \in A$ with $a \prec x$, then there exists a unitary $u$ such that

\[ a = E_A(\{uxu^*\}). \]

In particular, if $x$ is a projection with $\tau(x) = 1/\sqrt{2}$, $a = 1/\sqrt{2}$, there exists $u$ such that

\[ E(q) = a, \quad \text{where } q = uxu^*. \]

Corollary (Bhat-Ravichandran, 2011)

Let $A$ be a masa in the $II_1$-factor $M$. If $x \in M^{sa}$

\[ E_A(\{uxu^* : u \in \mathcal{U}(M)\}) = \{a \in A : a \prec x\} \]
Other recent results:

**Theorem (Dykema-Fang-Hadwin-Smith, 2011)**

Let $A \subset M$ be a masa in the $\text{II}_1$-factor $M$. Then the Carpenter’s Theorem holds in any of the following situations:

1. $M = L(F_S), |S| > |\mathbb{N}|$, $A$ any masa;
2. $M$ is not singly generated, $A$ any masa;
3. $A^\omega \subset M^\omega$, $A, M$ arbitrary;
4. $M = L(F_n)$, $A$ generated by $n-1$ or less of the generators;
5. $M = L(F_n)$, $A$ the radial masa;
6. $A \otimes A \subset R \otimes R$, where $R$ is the hyperfinite factors and $A$ is Cartan.
More recent results: $\mathcal{II}_\infty$ factors.

**Theorem (Argerami-Massey, 2011)**

Let $M$ be a $\mathcal{II}_\infty$ factor, $A \subset M$ a masa. Then

$$E_A(\mathcal{U}_M(b))^\mathcal{T} = \{ a \in A^{sa} : a \prec b \}.$$

Here $\mathcal{T}$ is the *measure topology*, i.e. the linear topology given by the neighbourhoods of $0 \in M$.

$$V(\varepsilon, \delta) = \{ r \in M : \exists p \in \mathcal{P}(M), \|rp\| < \varepsilon, \tau(p^\perp) < \delta \}$$
Thanks!