Renormalization group approach to singular perturbation theory for nonlinear PDEs

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Purpose

- Unification of different singular perturbation and asymptotic analysis methods for differential equations.
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- Unification of different singular perturbation and asymptotic analysis methods for differential equations.
- Investigate the rigorous application of the renormalization group method to (singular) perturbation theory for nonlinear partial differential equations (infinite-dimensional systems) in the *continuum* and in the presence of space-time *resonances*. 
Methods

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- The Coifman-Meyer theorem for Fourier multipliers to handle space-time resonances.
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- The Coifman-Meyer theorem for Fourier multipliers to handle space-time resonances.
- Fractional integration and dispersive estimates.
Some history

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- Applied to partial differential equations on **bounded intervals** with **periodic** boundary conditions. [Temam, Moise, Ziane, Petcu](2001-2005)
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u \in X$, a Banach space.
$A$ is a generator of a (continuous) semigroup on $X$.
We consider
\[ \partial_t u = Au + \epsilon f(u), \]
$f(u)$ is a polynomial nonlinearity, and $\epsilon \ll 1$.
Equivalent to studying
\[ \partial_\tau u = \frac{1}{\epsilon} Au + f(u) \]
after a simple rescaling.
Naive asymptotic expansion

Weak solution by the Duhamel formula

\[ u(t) = e^{A(t-t_0)}u_{t_0} + \epsilon e^{At} \int_{t_0}^{t} ds \ e^{-As} f(u(s)). \]

Naive perturbative expansion in \( \epsilon \) gives

\[ u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots, \]

where

\[ u_0(t) = e^{A(t-t_0)}u_{t_0}, \]

\[ u_1(t) = e^{At} \int_{t_0}^{t} ds \ e^{-As} f(u_0(s)) \]

\[ u_2(t) = e^{At} \int_{t_0}^{t} ds \ e^{-As} \nabla u f(u_0(s)) \cdot u_1(s) \]

\[ \cdots \]
Nonlinear term

\[ e^{-As} f(e^{As} v) = f_{\text{res}}(v) + f_{\text{osc}}(s, v) \]

\[ f_{\text{res}} \equiv \text{resonance part} \]
\[ f_{\text{osc}} \equiv \text{oscillatory part} \]
Secular term

Naive expansion

\[ u(t) = e^{A(t-t_0)}u_{t_0} + \epsilon(t - t_0)e^{At}f_{res}(e^{-At_0}u_{t_0}) \\
+ \epsilon e^{At}F_{osc}(t, e^{-At_0}u_{t_0}) + O(\epsilon^2). \]

Here

\[ F_{osc}(t, \nu) := \int_{t_0}^{t} ds \ f_{osc}(s, \nu) \]

Break down of naive expansion

Since the second term in the expansion (secular term) grows with \( t \), the naive perturbative expansion will break down with time!
RG method

The purpose of the CGO-method is to renormalize this term so that the main contribution coming from the secular term is taken into account.
Algorithm

\( \text{RG}_1 : X \rightarrow A \)

Map space of solutions to space of asymptotic expansions by expanding (formally) in a naive perturbation series.

\( \text{RG}_2 : A \rightarrow A \)

Do a transformation in the resulting asymptotic series such that all bounded time-independent terms are absorbed in the initial condition.

\( \text{RG}_3 : A \rightarrow X \)

Go back to the space of solutions using the RG condition. This involves setting the derivative of the asymptotic series obtained in \( \text{RG}_2 \) with respect to the initial time \( t_0 \) to zero.
Renormalization group condition

RG condition

\[ \partial_t W = \epsilon f_{\text{res}}(W), \]

with \( W(t_0) = e^{-At_0}u_{t_0} \).
Renormalization group condition

RG condition

\[ \partial_t W = \epsilon f_{\text{res}}(W), \]

with \( W(t_0) = e^{-A t_0} u_{t_0} \).

First order approximate solution

\[ \bar{u}(t) := e^{At} \{ W(t) + \epsilon F_{\text{osc}}(t, W(t)) \}. \]
Differentiating with respect to $t$ gives

$$\partial_t \bar{u} = A\bar{u} + \epsilon f(\bar{u}) + R_\epsilon(\bar{u}, W), \quad \bar{u}(t_0) = u_{t_0},$$

where

$$R_\epsilon(\bar{u}, W) = \epsilon(f(e^{At} W(t)) - f(\bar{u}))$$

$$+ \epsilon^2 e^{At} \nabla W F_{osc}(t, W(t)) f_{res}(W(t)).$$
Purpose of analysis

Show that $\bar{u}$ is an approximation of $u$ in a suitable Banach space, up to a certain time ($O(\frac{\log \epsilon}{\epsilon})$).
Normal form theory, when applicable, involves a near identity transformation on $X$ rather than the space of asymptotic expansions $A$. On the other hand, RG involved a near identity transformation that is applied to the initial condition in the asymptotic expansion.
Relationship to NF theory

NF theory

Normal form theory, when applicable, involves a near identity transformation on $X$ rather than the space of asymptotic expansions $A$. On the other hand, RG involved a near identity transformation that is applied to the initial condition in the asymptotic expansion.

Main advantages of RG over NF

1. Secular term identified by easy inspection in RG method.
2. RG applies to wide variety of (singular) perturbation theory, even in cases where NF theory is not developed.
Quadratic NLS in 3 spatial dimensions

\[ i\partial_t u = -\Delta u - \epsilon u^2 \]
A concrete example

Quadratic NLS in 3 spatial dimensions

\[ i \partial_t u = -\Delta u - \epsilon u^2 \]

Here

\[ A = i\Delta, \quad f(u) = iu^2. \]
Well-posedness for small initial data (Shatah et al, 2008)

\[ \| u \|_{\mathcal{B}_{\tau}} = \| u \|_{L^\infty([t_0, t_0 + \tau]; L^2(\mathbb{R}^3))} + \| e^{-i \Delta t} u \|_{L^\infty([t_0, t_0 + \tau]; L^\infty(\mathbb{R}^3))} + \left\| \frac{\chi}{\log t} e^{-i \Delta t} u \right\|_{L^\infty([t_0, t_0 + \tau]; L^2(\mathbb{R}^3))} + \left\| \frac{\chi^2}{\sqrt{t}} e^{-i \Delta t} u \right\|_{L^\infty([t_0, t_0 + \tau]; L^2(\mathbb{R}^3))} + \left\| t^{3/2} u \right\|_{L^\infty([t_0, t_0 + \tau]; L^\infty(\mathbb{R}^3))} \]
Main result

**Theorem**

Suppose

\[ \phi_0 := \| e^{i \Delta (t - t_0)} u_{t_0} \|_{B_\infty} < \infty. \]

Then there exists \( \epsilon_0 > 0 \), that depends on \( \phi_0 \), such that, for all \( |\epsilon| < \epsilon_0 \) and \( \delta \in (0, 1) \),

\[ \| u - \bar{u} \|_{L^\infty([t_0, t_0 + \delta \frac{\log |\epsilon|}{\epsilon \phi_0 \epsilon})]; L^2(\mathbb{R}^3)) < C \epsilon^{1-\delta} \]

for some positive constant \( C \) that is independent of \( \epsilon \) and \( \delta \).
Sketch of the proof

Approximate solution

\[
e^{-i\Delta s} f(e^{i\Delta s} u) = \]
\[
i \mathcal{F}^{-1} \int d\xi e^{is(k^2-\xi^2-(k-\xi)^2)} \hat{u}(\xi) \hat{u}(k - \xi)
\]
\[
\Rightarrow
\]
\[
f_{\text{res}}(u) = i \hat{u}(0) u, \]
\[
f_{\text{osc}}(u) = i \mathcal{F}^{-1} \int_{\xi \neq k} d\xi e^{is(k^2-\xi^2-(k-\xi)^2)} \hat{u}(\xi) \hat{u}(k - \xi).
\]
RG condition

\[ \partial_t W(t, x) = i \epsilon \hat{W}(t, 0) W(t, x) \]

with \( W(t_0) = e^{-i\Delta t_0} u_{t_0} \).
Lemma

Suppose that $\phi_0 < \infty$. Then

$$\sup_{s \in [t_0, t_0 + \frac{1}{e\phi_0\epsilon}]} \| e^{i\Delta t} W(s) \|_{L^\infty} \leq e\phi_0.$$
Lemma

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Prove the lemma by a boot-strap argument and Gronwall Lemma.
Lemma

Suppose that $\phi_0 < \infty$. Then

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Remark

Note that $W(t, x) = e^{i\epsilon \int_0^t ds \hat{W}(s, 0)} e^{-i\Delta t_0} u_{t_0}(x)$. 
Approximate solution

\[ \bar{u}(t) = e^{i \Delta t} W(t) + \epsilon i \int_{t_0}^{t} ds F^{-1} \left( \int_{\xi \neq k} d\xi e^{i(k^2 - \xi^2 - (k - \xi)^2)s} \hat{W}(t, \xi) \hat{W}(t, k - \xi)) \right), \]

\[ F_{osc}(t, W(t)) \]
Introduce the Banach space $C_\tau$ with norm

$$
\| u \|_{C_\tau} := \| u \|_{L^\infty([t_0, t_0+\tau]; L^2(\mathbb{R}^3))} + \| t^{3/2} u \|_{L^\infty([t_0, t_0+\tau]; L^\infty(\mathbb{R}^3))}.
$$

Using the Coifman-Meyer theorem and fractional integration, one has

$$
\| e^{i \Delta t} F_{osc}(t, W(t)) \|_{C_1 \frac{1}{e^{\phi_0 \epsilon}}} \leq C.
$$

This is a key technical estimate in the proof of the following lemma.

**Lemma**

Suppose that $\phi_0 < \infty$. Then $\bar{u} \in C \frac{1}{e^{\phi_0 \epsilon}}$. 
Proof of main result

Proposition

Suppose that $\phi_0 < \infty$. Then there exists $\epsilon_0 > 0$ such that, for all $|\epsilon| < \epsilon_0$,

$$\|u(t) - \bar{u}(t)\|_{L^\infty([t_0, t_0 + \frac{1}{\epsilon \phi_0 \epsilon}; L^2(\mathbb{R}^3))] < C \epsilon,$$

for some positive constant $C$ that is independent of $\epsilon$.

Iterate to get the main result.
Coifman-Meyer Theorem

Consider the Fourier multiplier \( m(k, \xi) \) and the associated operator \( T_m \) defined by

\[
T_m(f, g) := \mathcal{F}^{-1} \int d\xi \ m(k, \xi) \hat{f}(\xi) \hat{g}(k - \xi)
\]

where \( \mathcal{F} \) is the inverse Fourier transform. Suppose that the Fourier multiplier \( m \) satisfies

\[
|\partial^\alpha_k \partial^\beta_\xi m(k, \xi)| \leq \frac{C}{(|k| + |\xi|)|\alpha| + |\beta|}
\]

for sufficiently many multi-indices \((\alpha, \beta)\). Then the operator

\[
T_m : L^p \times L^q \to L^r
\]

is bounded for

\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{r}, \quad 1 < p, q \leq \infty, \quad 0 < r < \infty.
\]
Fractional integration

Let $\Lambda^\alpha := (-\Delta)^{\frac{\alpha}{2}}$ and $\Lambda_t^\alpha := (\frac{1}{t} - \Delta)^{\frac{\alpha}{2}}$, $t > 0$. Then the following holds.

(i) If $\alpha \geq 0$ and $1 < p, q < \infty$, \( \frac{1}{q} - \frac{1}{p} = \frac{\alpha}{3} \), then

\[
\|\Lambda^{-\alpha} f\|_{L^p} \leq C \|f\|_{L^q}
\]

for some constant $C > 0$.

(ii) If $\alpha \geq 0$ then

\[
\|\Lambda^{-\alpha} f\|_{L^\infty} \leq C \|f\|_{L^{\frac{3}{\alpha}, 1}}
\]

for some constant $C > 0$, where $L^{p, q}$ is the standard Lorenz space.

(iii) If $\alpha \geq 0$ and $1 \leq p, q \leq \infty$, $0 \leq \frac{1}{q} - \frac{1}{p} < \frac{\alpha}{3}$, then

\[
\|\Lambda_t^{-\alpha} f\|_{L^p} \leq Ct^{\frac{\alpha}{2} + \frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^q}
\]

for some constant $C > 0$. 

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