Orbits of conditional expectations

M. Argerami and D. Stojanoff*

Abstract

Let $N \subseteq M$ be von Neumann algebras and $E : M \to N$ a faithful normal conditional expectation. In this work it is shown that the similarity orbit $\mathcal{S}(E)$ of E by the natural action of the invertible group of G_M of M has a natural complex analytic structure and the map given by this action: $G_M \to \mathcal{S}(E)$ is a smooth principal bundle. It is also shown that if N is finite then $\mathcal{S}(E)$ admits a Reductive Structure. These results were known previously under the conditions of finite index and $N' \cap M \subseteq N$, which are removed in this work. Conversely, if the orbit $\mathcal{S}(E)$ has an Homogeneous Reductive Structure for every expectation defined on M, then M is finite. For every algebra M and every expectation E, a covering space of the unitary orbit $\mathcal{U}(E)$ is constructed in terms of the connected component of 1 in the normalizer of E. Moreover, this covering space is the universal covering in any of the following cases: 1) M is a finite factor and $Ind(E) < \infty$; 2) M is properly infinite and E is any expectation; 3) E is the conditional expectation onto the centralizer of a state. Therefore, in those cases, the fundamental group of $\mathcal{U}(E)$ can be characterized as the Weyl group of E.

1. Introduction.

Let M be a von Neumann algebra with group of invertible elements G_M and unitary group $\mathcal{U}_M$. Denote by $\mathcal{E}(M)$ the space of faithful normal conditional expectations defined on M and $\mathcal{B}(M)$ the algebra of bounded linear operators on M. Consider the action

$$L : G_M \times \mathcal{B}(M) \to \mathcal{B}(M)$$

given by

$$L_g(T) = gT(g^{-1} \cdot g)g^{-1}, \quad g \in G_M,\ T \in \mathcal{B}(M).$$

*Research partially supported by CONICET, ANPCYT and UBACYT (Argentina)

Let $E \in \mathcal{E}(M)$ be a conditional expectation. Define the unitary orbit of E

$$\mathcal{U}(E) = \{L_u(E) : u \in \mathcal{U}_M\} \subseteq \mathcal{E}(M). \tag{1}$$

considered with the quotient topology induced by the norm topology of $\mathcal{U}_M$. So we have a natural fibration

$$\Pi_E : \mathcal{U}_M \to \mathcal{U}(E) \quad \text{given by} \quad \Pi_E(u) = L_u(E),\ u \in \mathcal{U}_M \tag{2}$$

The objective of this work is the study of the homotopy groups and the differential geometry of the orbit $\mathcal{U}(E)$ or, more precisely, of the fibration Π_E. Several results in this sense appear in the works [4], [3] and [21], mainly under two very restrictive hypothesis: finite index condition for E and the condition $E(M)' \cap M \subseteq E(M)$ for the inclusion of the algebras. Also in [5] a complete study of this problems is made in the case when E is a state.

In order to study the homotopy type of these orbits we construct a covering space over each orbit $\mathcal{U}(E)$ whose group of covering transformations is the so called Weyl group of the expectation E. To describe this structure we need the following definitions:

At the level of the unitary group $\mathcal{U}_M$ of M, the isotropy group of the action L, i.e. $\Pi_E^{-1}(E)$ is a very well known group usually called the *normalizer* of E. It has already been studied, between other authors, by A. Connes ([10]) and Kosaki ([18]) in relation with crossed product inclusions of algebras (see also [7]). We shall denote the normalizer of E by

$$\mathcal{N}_E = \{\ u \in \mathcal{U}_M\ :\ E(uxu^*) = uE(x)u^*, x \in M\ \}. \tag{3}$$

Let $N = E(M)$. Then N is a von Neumann subalgebra of M. Consider also the von Neumann algebra

$$M_E = \{x \in N' \cap M\ :\ E(xm) = E(mx) \ \text{ for all }\ m \in M\}, \tag{4}$$

usually denoted as the centralizer of E (see [9] or [14]). In [7] it is shown that the connected component of 1 in $\mathcal{N}_E$ is the group

$$\mathcal{H}_E = \mathcal{U}_N \cdot \mathcal{U}_{M_E} = \{vw : v \in \mathcal{U}_N \ \text{ and }\ w \in \mathcal{U}_{M_E}\}, \tag{5}$$

which is a close, open and invariant subgroup of $\mathcal{N}_E$. Then the set of connected components of $\mathcal{N}_E$ is a discrete group called the Weyl group of E:

$$W(E) = \pi_0(\mathcal{N}_E) \simeq \mathcal{N}_E/\mathcal{H}_E, \tag{6}$$

which has several characterizations in very different contexts (see [18], [7] and [8]).

We show that, for any von Neumann algebra M and any $E \in \mathcal{E}(M)$, the space $\mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E$ and its natural projection onto $\mathcal{U}(E) \simeq \mathcal{U}_M/\mathcal{N}_E$ (see diagram (9)) defines a covering map whose group of covering transformations can be identified with the Weyl group $W(E)$ (Theorem 2.3).

In all examples we know, $\mathcal{X}(E)$ is actually the *universal* covering for $\mathcal{U}(E)$ and therefore the fundamental group $\pi_1(\mathcal{U}(E), E)$ coincides with the Weyl group $W(E)$. We conjecture that this is true for all von Neumann algebras M and all conditional expectations $E \in \mathcal{E}(M)$. In Theorem 2.6 we show that if any of the following conditions hold:

1. M is properly infinite,
2. M is finite, $\dim \mathcal{Z}(M) < \infty$ and E has finite index,
3. $E = E_\varphi \in \mathcal{E}(M, M_\varphi)$ is the canonical expectation associated to a faithful normal state φ of M, i.e. M_φ is the centralizer of φ and $\varphi \circ E = \varphi$,

then $\mathcal{X}(E)$ is simply connected, so that it is the universal covering for the orbit $\mathcal{U}(E)$. Consequently,

$$\pi_1(\mathcal{U}(E), E) \simeq W(E).$$

In order to study the differential geometry of the orbit of an expectation E we consider the whole similarity orbit

$$\mathcal{S}(E) = \{L_g(E) : g \in G_M\} \subseteq \mathcal{B}(M) \tag{7}$$

and the fibration (with the same name as its restriction to $\mathcal{U}_M$),

$$\Pi_E : G_M \to \mathcal{S}(E) \quad \text{given by} \quad \Pi_E(g) = L_g(E),\ g \in G_M. \tag{8}$$

Note that $L_g(E)$ is not necessarily a conditional expectation for all $g \in G_M$. Nevertheless we prefer to use this setting, since the group G_M is a complex analytic Banach Lie group and then the orbit $\mathcal{S}(E)$ can be given a complex analytic manifold structure. In any case, all the geometrical results obtained for $\mathcal{S}(E)$ are also true for the unitary orbit $\mathcal{U}(E)$ by just replacing "complex analytic" by "real analytic".

In order to study the differential geometry of similarity orbits we need to generalize several results of [7] mentioned before to the invertible groups setting. This task is made in section 3, where the connected component of the isotropy group I_E of the action L at E is characterized (see Proposition 3.3) and the new Weyl group which naturally appears is shown to be the same group as the "old" one (see Theorem 3.5).

In section 4 we first show that $\mathcal{S}(E)$ can be always be given a unique complex analytic differential structure such that the map Π_E of equation (8) becomes a submersion (Theorem 4.8). The key tool is the construction, in the style of [9], of a conditional expectation $F \in \mathcal{E}(M)$ onto the centralizer M_E which commutes with E. This allows us to get a complement in M for the subspace $N + M_E$, which can be naturally identified with the tangent space of $\mathcal{S}(E)$ at E.

Then we show that if $N = E(M)$ is a finite von Neumann algebra, $\mathcal{S}(E)$ has a unique structure of Homogeneous Reductive Space (HRS) (see Definition 4.10 and Proposition 4.13). This family of HRS's are of geometrical interest. Indeed, maybe the most general families of examples of infinite dimensional HRS's modeled in operator algebras are studied in [1] and [3]. But all these examples can be represented as the quotient of the unitary (or invertible) group of an algebra M by the unitary (or invertible) group of some subalgebra. This situation does not happen in the case of the orbit of a conditional expectation. Indeed, the isotropy group I_E can be big enough even to generate the whole algebra M, while $\mathcal{S}(E) \simeq G_M/I_E$. Moreover, also the map $\Pi_0 : G_M \to G_M/\mathcal{Z}_E$ $(= \mathcal{Y}(E))$, where $\mathcal{Z}_E$ is the connected component of 1 in I_E, defines a HRS if N is finite. Actually (see Theorem 4.8) this is the way to show the HRS structure of $\mathcal{S}(E)$, since $\mathcal{Y}(E)$ is a covering space for $\mathcal{S}(E)$ and therefore they are locally homeomorphic (and diffeomorphic). But neither $\mathcal{Y}(E)$ can be represented as a quotient as before (if $N \not\subseteq M_E$ and $M_E \not\subseteq N$), since, by Proposition 4.6, $\mathcal{Z}_E = G_{M_E} \cdot G_N$ and this is not the invertible group of any subalgebra of M.

In the end of section 4 we show that the existence of HRS structures for any expectation $E \in \mathcal{E}(M)$ forces M to be a finite von Neumann algebra (Theorem 4.17).

2. The universal covering of $\mathcal{U}(E)$.

Let $N \subseteq M$ be von Neumann algebras. From now on we shall denote by $\mathcal{E}(M, N)$ the space of faithful normal conditional expectations $E : M \to N$. Let $E \in \mathcal{E}(M, N)$. We recall the definitions of the sets $\mathcal{U}(E)$, $\mathcal{N}_E$, M_E and $\mathcal{H}_E$ (see equations (1), (3), (4) and (5), respectively) associated with E. Consider the space $\mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E$, with the quotient topology of the norm topology of $\mathcal{U}_M$ and denote by Π_0 the projection form $\mathcal{U}_M$ onto $\mathcal{X}(E)$. The situation we shall study is the following: we have a commutative diagram

$$\begin{array}{ccc} \mathcal{U}_M & \xrightarrow{\Pi_0} & \mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E \\ & \Pi_E \searrow & \Big\downarrow \Phi \\ & & \mathcal{U}(E) \simeq \mathcal{U}_M/\mathcal{N}_E \end{array} \qquad (9)$$

where the map Φ is defined by $\Phi(u\mathcal{H}_E) = \Pi_E(u) \sim u\mathcal{N}_E$, $u \in \mathcal{U}_M$. In [4] it was shown that when $N' \cap M \subseteq N$, and the Jones index of E is finite, calling e the Jones projection of E, then its M-unitary orbit

$$\mathcal{U}_M(e) = \{ueu^* : u \in \mathcal{U}_M\} \simeq \mathcal{U}_M/\mathcal{U}_N,$$

is a covering space for $\mathcal{U}(E)$. Note that, under the mentioned assumptions, $\mathcal{U}_M(e) \simeq \mathcal{X}(E)$, since both spaces can be identified with $\mathcal{U}_M/\mathcal{U}_N$ and the fact, showed in [4], that the quotient topology and the norm topology coincide on $\mathcal{U}_M(e)$.

In what follows we will show, removing both hypothesis appearing in [4], that the map Φ is always a covering map, and that $\mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E$ is a covering space for $\mathcal{U}(E)$, with group of covering transformations $W(E)$. Moreover, in several cases (see 2.6) Φ is the universal covering of $\mathcal{U}(E)$ and, in particular, $\pi_1(\mathcal{U}(E)) \simeq W(E)$.

Note that the Weyl group $W(E) = \mathcal{N}_E/\mathcal{H}_E$, being included in $\mathcal{X}(E)$, has a natural action on $\mathcal{X}(E)$ given by right multiplication. This action is well defined because $\mathcal{H}_E$ is a normal subgroup of $\mathcal{N}_E$.

Proposition 2.1 *Let $N \subseteq M$ be von Neumann algebras and $E \in \mathcal{E}(M,N)$ a faithful normal conditional expectation. Then, with the notations of diagram (9),*

1. *The map Φ is continuous.*
2. *For any $u \in \mathcal{U}_M$, the fibre by Φ of $L_uE \in \mathcal{U}(E)$ is precisely the orbit of $\Pi_0(u) \in \mathcal{X}(E)$ by the action of $W(E)$.*
3. *The unitary orbit $\mathcal{U}(E)$ is homeomorphic with $\mathcal{X}(E)/W(E)$ (i.e. the space of orbits by the action of $W(E)$ in $\mathcal{X}(E)$), both considered with the quotient topology.*

Proof. Items 1 and 2 follow immediately by looking at the commutative diagram (9) and using the fact that $\Phi^{-1}(E) = W(E)$. Let $\rho : \mathcal{X}(E) \to \mathcal{X}(E)/W(E)$ be the canonical projection. Consider the map $\bar{\Phi} : \mathcal{X}(E)/W(E) \to \mathcal{U}(E)$ given by $\bar{\Phi}(\rho(h)) = \Phi(h)$, $h \in \mathcal{X}(E)$. Then $\bar{\Phi}$ is the desired homeomorphism. Indeed, it is clear that $\bar{\Phi}$ is well defined and is bijective. $\bar{\Phi}$ is also continuous, since $\Phi = \bar{\Phi} \circ \rho$ is continuous. On the other hand, let U be an open set in $\mathcal{X}(E)/W(E)$. By considering the full commutative diagram

$$\begin{array}{ccccc} \mathcal{U}_M & \xrightarrow{\Pi_0} & \mathcal{X}(E) = \mathcal{U}_M/\mathcal{H}_E & \xrightarrow{\rho} & \mathcal{X}(E)/W(E) \\ & {\scriptstyle \Pi_E} \searrow & {\scriptstyle \Phi} \Big\downarrow & \swarrow {\scriptstyle \bar{\Phi}} & \\ & & \mathcal{U}(E) = \mathcal{U}_M/\mathcal{N}_E & & \end{array} \tag{10}$$

and the fact that Π_E is an open map, it is clear that $\bar{\Phi}(U)$ is open in $\mathcal{U}(E)$. $\square$

Remark 2.2 In order to show that the map Φ defined in diagram (9) is a covering map we shall use the following well known result of algebraic topology (see, for instance, Chapter 1 of [13]):

Let X be a locally pathwise connected and connected topological space, and G a groups of homeomorphisms of X that operates properly discontinuously (i.e. for each $x \in X$ there exists an open set V_x such that $V_x \cap g(V_x) =$ for every $g \in G$, $g \neq e$). Consider the map $p : X \to X/G$. Then X is a covering space, for X/G with covering map p and group of covering transformations G, and $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, p(x_0))$.

Theorem 2.3 *Let $N \subseteq M$ be von Neumann algebras and $E \in \mathcal{E}(M, N)$ a faithful normal conditional expectation. Then, with the notations of diagram (9), the space $\mathcal{X}(E)$ is a covering space for $\mathcal{U}(E)$, with covering map Φ and group of covering transformations $W(E)$.*

Proof. By the previous remark, it suffices to show that $W(E)$ operates properly discontinuously on $\mathcal{X}(E)$. Fix $u \in \mathcal{U}_M$ and consider the open set

$$W_u = \{w \in \mathcal{U}_M : \|w - u\| < 1/2\}.$$

For each element $k \in W(E)$, we choose some $u_k \in \mathcal{N}_E$ such that $\Pi_0(u_k) = u_k\mathcal{H}_E = k$. Since the map Π_0 is open, we can consider the open set

$$V_u = \Pi_0(W_u) \subseteq \mathcal{X}(E).$$

Note that, for $k \in W(E)$, $V_u k = \Pi_0(W_u u_k)$. In order to prove that the action of $W(E)$ in $\mathcal{X}(E)$ is properly discontinuous, we just have to see that $V_u \cap V_u k = \phi$ for every $1 \neq k \in W(E)$. Suppose that this is not true. Then, for some $1 \neq k \in W(E)$, there exist $w_1, w_2 \in W_u$ and $z \in \mathcal{H}_E$ such that $w_1 u_k = w_2 z$. Then

$$w_1^* w_2 = u_k z^* \in \mathcal{N}_E \setminus \mathcal{H}_E$$

But, since $w_1, w_2 \in W_u$,

$$\|w_1^* w_2 - 1\| = \|w_2 - w_1\| < 1.$$

This implies that $w_1^* w_2 \in \mathcal{H}_E$ (see [7] or the proof of Proposition 3.3 below), a contradiction. □

Corolary 2.4 The group $\Psi_*(\pi_1(\mathcal{X}(E)))$ is a normal subgroup of $\pi_1(\mathcal{U}(E))$ and we have the following isomorphism:

$$\pi_1(\mathcal{U}(E))/\Psi_*(\pi_1(\mathcal{X}(E))) \simeq W(E)$$

Proof. By Proposition 2.1, we know that the fibre $\Psi^{-1}(E)$ equals $W(E)$, and the assertion follows by using the homotopy exact sequence induced by the covering map Ψ. □

Remark 2.5 Let φ be a faithful normal state of the von Neumann algebra M. In [5] Andruchow and Varela show that the unitary orbit of φ:

$$\mathcal{U}(\varphi) = \{\varphi(u^* \cdot u) : u \in \mathcal{U}_M\}$$

is simply connected. Therefore the unitary group of the centralizer M_φ of φ coincides with the normalizer $\mathcal{N}_\varphi$ of φ, considered as a conditional expectation. Then the covering space

$$\mathcal{X}(\varphi) = \mathcal{U}_M/\mathcal{U}_{M_\varphi} = \mathcal{U}_M/\mathcal{N}_\varphi \simeq \mathcal{U}(\varphi)$$

and $\mathcal{U}(\varphi)$ is its own universal covering.

Moreover, if $E_\varphi \in \mathcal{E}(M, M_\varphi)$ is the canonical expectation such that $\varphi \circ E_\varphi = \varphi$, then $\mathcal{U}(\varphi) \simeq \mathcal{X}(E_\varphi)$ and so it is the universal covering for $\mathcal{U}(E_\varphi)$. Indeed, since

$$\mathcal{X}(E_\varphi) = \mathcal{U}_M/\mathcal{U}_{M_\varphi}\mathcal{U}_{M_{E_\varphi}} \quad \text{and} \quad \mathcal{U}(\varphi) \simeq \mathcal{U}_M/\mathcal{U}_{M_\varphi},$$

it suffices to show that $M_{E_\varphi} \subseteq M_\varphi$. But this is apparent by the definition of M_{E_φ} (see equation 4) and the fact that $\varphi \circ E_\varphi = \varphi$. This gives a large class of conditional expectations for which the covering space $\mathcal{X}(E)$ is the universal covering. We shall extend this class in the following Theorem.

Theorem 2.6 *Let M be a von Neumann algebra, $E \in \mathcal{E}(M)$, and suppose that any of the following conditions hold:*

1. *M is properly infinite.*
2. *M is a II_1 factor and E has finite index.*
3. *$E = E_\varphi \in \mathcal{E}(M, M_\varphi)$ is the canonical expectation associated to a faithful normal state φ of M as in Remark 2.5.*

Then $\mathcal{X}(E)$ is simply connected, so that it is the universal covering for the orbit $\mathcal{U}(E)$. Consequently,

$$\pi_1(\mathcal{U}(E), E) \simeq W(E).$$

Proof. Consider the fibre bundle

$$\Pi_0 : \mathcal{U}_M \to \mathcal{U}_M/\mathcal{H}_E = \mathcal{X}(E). \tag{11}$$

Recall that a fibre bundle gives rise to an exact sequence of homotopy groups. In our case, the bundle Π_0 yields the exact sequence

$$\ldots \to \pi_2(\mathcal{X}(E)) \to \pi_1(\mathcal{H}_E) \xrightarrow{i_*} \pi_1(\mathcal{U}_M) \to \pi_1(\mathcal{X}(E)) \to \pi_0(\mathcal{H}_E) = 0, \qquad (12)$$

where 1 is taken as base point for the homotopy groups of the unitary groups and $[1]_{\mathcal{X}(E)} = \mathcal{H}_E$ is the base point for $\mathcal{X}(E)$. Here i_* denotes the homomorphism induced by the inclusion $i : \mathcal{H}_E \hookrightarrow \mathcal{U}_M$. We can then use results by Handelmann [15] and Schröeder [27] on computing the homotopy group of the unitary group of a von Neumann algebra.

Case (1.) It follows by appealing to the homotopy exact sequence (12), and the fact [15] that $\mathcal{U}_M$ has trivial π_1 group if M is properly infinite.

Case (2.): Since M is a II_1 factor and $\mathrm{Ind}(E) < \infty$ it is known (see [25]) that $N = E(M)$ is also of type II_1 and $\dim \mathcal{Z}(N) < \infty$. Let us recall the following results (see [5], [15] and [27]):

1. If M is a von Neumann algebra of type II_1, then $\pi_1(\mathcal{U}_M)$ is isomorphic to (the additive group) $\mathcal{Z}(M)_{sa}$ of selfadjoint elements in $\mathcal{Z}(M)$.

2. Let $j : \mathcal{U}_N \to \mathcal{U}_M$ be the inclusion map. Then the image of the homomorphism $j_* : \pi_1(\mathcal{U}_N) \to \pi_1(\mathcal{U}_M) \simeq \mathcal{Z}(M)_{sa}$ is equal to the additive group generated by the set $\{tr(p) : p \text{ projection in } N\}$, where tr is the center valued trace of M.

In our case $\pi_1(\mathcal{U}_M) \simeq I\!R$. Let $k : \mathcal{U}_N \to \mathcal{H}_E$ be the inclusion map. Clearly $i_* \circ k_* = j_*$, where i_* is the map of equation (12). Then $j_*(\pi_1(\mathcal{U}_N)) \subseteq i_*(\pi_1(\mathcal{H}_E))$. Let $p \in \mathcal{Z}(N)$ be a minimal projection. Then pNp is a II_1 factor and the trace of projections only in pNp generate the additive group $I\!R$. Then i_* is surjective and $\pi_1(\mathcal{X}(E))$ must be trivial by the homotopy exact sequence (12). Case (3.) follows from [5] and Remark 2.5 □

Remark 2.7 With the same techniques used in the proof of the previous Theorem, it can be also shown that $\mathcal{X}(E)$ is simply connected if $\mathrm{Ind}(E) < \infty$, M is finite and $\dim \mathcal{Z}(M) < \infty$.

Example 2.8 Let M be a von Neumann algebra and $p \in M$ be a projection. Then p determines the conditional expectation $E_p : M \to N = \{p\}' \cap M$ given by

$$E_p(x) = pxp + (1-p)x(1-p), \quad x \in M.$$

Denote by $\mathcal{U}(p) = \{upu^* : u \in \mathcal{U}_M\}$ the unitary orbit of p, which is a connected component of the Grassmannians of M. Then

$$\mathcal{U}(p) \simeq \mathcal{X}(E_p)$$

in the sense that both spaces are homeomorphic to $\mathcal{U}_M/\mathcal{U}_N$, since $N' \cap M \subseteq N$ and so $\mathcal{H}_{E_p} = \mathcal{U}_N$. Note that on $\mathcal{U}(p)$ we consider the norm topology as a subset of M (see [12] or [26]). Using Theorem 2.6 it is not difficult to show that the Grassmannian $\mathcal{U}(p)$ is always simply connected. Indeed, $\pi_1(\mathcal{U}(p))$ splits in the finite and the properly infinite parts of M and items 1 and 3 of 2.6 can by applied (see also [2]).

The Weyl group of E_p is trivial if $1-p \notin \mathcal{U}(p)$ and it has two elements if $1-p \in \mathcal{U}(p)$, since in this case, if $u \in \mathcal{U}_M$ verifies that $upu^* = 1-p$, then $L_u E_p = E_{1-p} = E_p$ and $\mathcal{N}_{E_p} = \mathcal{U}_N \cup u \cdot \mathcal{U}_N$.

A similar study can be done for systems of projections, i.e the n-tuples $P = (p_1, \ldots, p_n)$ of pairwise orthogonal projections such that $\sum p_i = 1$ (see [11]). Here also (by Theorem. 2.6) the joint unitary orbit of P is simply connected and it is homeomorphic to the space $\mathcal{X}(E_P)$ associated to the conditional expectation $E_P(x) = \sum p_i x p_i$, $x \in M$. The Weyl group is a subgroup of the permutation group S_n, determined by those projections in P such that are equivalent (and therefore unitary equivalent) in M.

3. The Weyl group, invertible case.

In [7], the Weyl group is defined in terms of the unitary group of the von Neumann algebra M. Our aim in this section is to extend the previous development to the case when the action over the conditional expectations is given not anymore by the unitaries but by the invertible elements.

Let us recall some definitions. Let M be a von Neumann algebra. We consider the action $L : G_M \times \mathcal{B}(M) \to \mathcal{B}(M)$ given by $L_g(T) = gT(g^{-1} \cdot g)g^{-1}$, $g \in G_M$, $T \in \mathcal{B}(M)$. Let $E \in \mathcal{E}(M)$ be a conditional expectation. Then, as we have already mentioned, $L_g(E)$ is not necessarily a conditional expectation for all $g \in G_M$, but we still consider the orbit of the expectation

$$\mathcal{S}(E) = \{L_g(E) : g \in G_M\} \quad \text{and the fibration} \quad \Pi_E : G_M \to \mathcal{S}(E)$$

given by $\Pi_E(g) = L_g(E)$, $g \in G_M$. The role played in the unitary case by the normalizer $\mathcal{N}_E = \{u \in \mathcal{U}_M : L_u(E) = E\}$ as the isotropy group of the action is now played by

$$I_E = \{g \in G_M : L_g(E) = E\}, \tag{13}$$

Let $N = E(M) \subseteq M$. Then N is a von Neumann algebra. Recall that the centralizer of E is the von Neumann algebra $M_E = \{x \in N' \cap M \ : \ E(xm) = E(mx)$ for all $m \in M\}$. Define the group

$$\mathcal{Z}_E = G_{M_E} \cdot G_N \subseteq I_E \tag{14}$$

Proposition 3.1 *Let M be a von Neumann algebra, $E \in \mathcal{E}(M)$ and consider the groups I_E and $\mathcal{Z}_E$ defined in equations (13) and (14). Then*

1. *If $g \in I_E$, then $gE(g^{-1}) = E(g^{-1})g \in M_E$.*
2. *If $g \in I_E$ and $E(g^{-1})$ is invertible, then $g \in \mathcal{Z}_E$.*
3. *$I_E \cap M^+ = \mathcal{Z}_E^+$.*

Proof. If $g \in I_E$, then $gE(g^{-1}) = gE(g^{-1}g^{-1}g) = E(g^{-1})g$. Let $N = E(M)$ and $b \in N$. Then

$$gE(g^{-1})b = gE(g^{-1}b) = gE(g^{-1}bg^{-1}g) = E(bg^{-1})g = bE(g^{-1})g,$$

so that $gE(g^{-1}) \in N' \cap M$. If $x \in M$, since $gNg^{-1} = N$,

$$E(gE(g^{-1})x) = E(E(g^{-1})gx) = E(g^{-1})E(gx) = E(g^{-1})gE(xg)g^{-1} = E(xgE(g^{-1})),$$

thus proving that $gE(g^{-1}) \in M_E$.

If $E(g^{-1})$ is invertible, then $g = gE(g^{-1}) \cdot E(g^{-1})^{-1} \in G_{M_E}G_N = \mathcal{Z}_E$ by 1. Finally, if $g \in I_E$ and $g > 0$, then $g^{-1} > 0$, and, as E is faithful, $E(g^{-1}) > 0$. Then $E(g^{-1}) \in G_N$, and by 2, $g \in \mathcal{Z}_E$. □

Lemma 3.2 *If $g \in G_M$ and $\|g - 1\| < \varepsilon < 1$, then*

$$\|g^{-1} - 1\| < \frac{\varepsilon}{1-\varepsilon}.$$

In particular, if $\varepsilon \leq 1/2$, then

$$\|g^{-1} - 1\| < 2\varepsilon.$$

Proof. Straightforward. □

Proposition 3.3 *Let M be a von Neumann algebra, $E \in \mathcal{E}(M)$ and consider the groups $\mathcal{Z}_E \subseteq I_E$ defined in equations (13) and (14). The group $\mathcal{Z}_E$ is open, closed and connected in I_E. Moreover, the connected component of I_E at any $u \in I_E$ is exactly $u \cdot \mathcal{Z}_E$.*

Proof. First of all we will show that $\mathcal{Z}_E$ is open at 1. Let $g \in I_E$ such that $\|g - 1\| < 1/2$. Then, by Lemma 3.2, we have that

$$\|g^{-1} - 1\| < 1, \quad \text{so} \quad \|E(g^{-1}) - 1\| < 1,$$

implying that $E(g^{-1})$ is invertible. So, by Proposition 3.1, $g \in \mathcal{Z}_E$.

If $h \in \mathcal{Z}_E$, by the preceding paragraph there is a neighborhood V of 1 such that $V \cap I_E \subseteq \mathcal{Z}_E$. Then hV is a neighborhood of h and, if $g \in hV$, then $h^{-1}g \in V$, so that $h^{-1}g \in \mathcal{Z}_E$. As $h \in \mathcal{Z}_E$, we have that $g \in \mathcal{Z}_E$, so $\mathcal{Z}_E$ is open. Clearly $g\mathcal{Z}_E$ is open for every $g \in I_E$, and as we can obtain I_E as a disjoint union of sets $g\mathcal{Z}_E$, all open, they are also closed. $\mathcal{Z}_E$ can be easily seen to be connected, since it is the product of two connected groups. The last assertion becomes now clear. □

We deduce that the group $\mathcal{Z}_E$ is closed, open and invariant in I_E. So we have a new Weyl group defined by

$$W_1(E) = \pi_0(I_E) \simeq I_E/\mathcal{Z}_E \tag{15}$$

Next we shall see that the new Weyl group agrees with the old (unitary) one. We first need the following Lemma:

Lemma 3.4 *With the above notations,*

$$\mathcal{H}_E = \mathcal{U}_{M_E}\mathcal{U}_N = \mathcal{N}_E \cap G_{M_E}G_N = \mathcal{N}_E \cap \mathcal{Z}_E.$$

Proof. Let $w \in \mathcal{N}_E \cap \mathcal{Z}_E$. Then $w = mn$, with $m \in G_{M_E}, n \in G_N$. Using the polar decomposition of m and n and the fact that $M_E \subseteq N' \cap M$, we have

$$w = v_m|m| \cdot v_n|n| = v_m v_n|m||n| = v_m v_n|mn| = v_n v_m|w| = v_n v_m,$$

where $v_m \in \mathcal{U}_{M_E}, v_n \in \mathcal{U}_N$, so that $w \in \mathcal{H}_E$. □

Now we have the technical tools we need to prove that both Weyl groups are the same.

Theorem 3.5 *Let M be a von Neumann algebra, $E \in \mathcal{E}(M)$. Then the Weyl group obtained by the unitary construction and the Weyl group obtained by the invertible construction are isomorphic (i.e. $W_1(E) = W(E)$).*

Proof. Let $\varphi : W(E) \to W_1(E)$ given by $\varphi([u]_{W(E)}) = [u]_{W_1(E)}$, for $u \in \mathcal{N}_E$. Then φ is well defined and it is an isomorphism. Indeed, good definition is clear since $\mathcal{H}_E \subseteq \mathcal{Z}_E$. By Lemma 3.4, φ is injective. Let $g \in I_E$. To see that the map φ is onto, We must find a unitary $u \in \mathcal{N}_E$ such that $[g]_{W_1(E)} = [u]_{W_1(E)}$.
Since $g \in I_E$, we have that $(g^*)^{-1} \in I_E$ (by adjoining and using that E is $*$-linear) and so $g^* \in I_E$, since I_E is a group. Therefore $g^*g \in I_E$ and, by Proposition 3.1, $g^*g \in \mathcal{Z}_E$. Then there exist $m \in M_E$, $n \in N$ with $g^*g = mn$. Using again the polar decompositions $m = v_m|m|$ and $n = v_n|n|$ with $v_m \in \mathcal{U}_{M_E}$ and $v_n \in \mathcal{U}_N$, we have that

$$g^*g = mn = v_m|m| \cdot v_n|n| = v_m v_n|m||n| = v_m v_n|mn| = v_m v_n|g^*g| = v_m v_n g^*g,$$

implying that $v_m v_n = 1$, so we can write $g^*g = mn$ with $m \in M_E^+$, $n \in N^+$. Then $|g| = m^{1/2}n^{1/2} \in \mathcal{Z}_E$. Using the polar decomposition of g, there is a unitary $u \in \mathcal{U}_M$ with $g = u|g|$. As $|g| \in \mathcal{Z}_E$ and $g \in I_E$, it follows that $u \in \mathcal{U}_M \cap I_E = \mathcal{N}_E$, so $[g] = [u]$. Finally, since both groups are discrete, the mentioned isomorphism is also a homeomorphism. □

Remark 3.6 Almost all the construction made in this paper can be extended trivially to C* algebras. Proposition 3.3 is the point were problems appear since the invertible group of a C* algebra need not to be connected.

4. Differential geometry of $\mathcal{S}(E)$.

In this section we shall consider only von Neumann algebras with separable predual, in order to assure the existence of faithful normal states.

Let $N \subseteq M$ be von Neumann algebras. $E \in \mathcal{E}(M, N)$ and $\mathcal{S}(E) = \{gE(g^{-1}\cdot g)g^{-1} : g \in G_M\}$. The differential geometry of the orbit $\mathcal{S}(E)$ has been already studied by Larotonda and Recht in [21], where it is assumed that $N' \cap M \subseteq N$. In this case they show that $\mathcal{S}(E)$ admits a differentiable structure and the map $\Pi_E : G_M \to \mathcal{S}(E)$ defines a reductive structure on $\mathcal{S}(E)$.

The aim of this section is to remove that hypothesis, and we shall show that the orbit $\mathcal{S}(E)$ can be always given a differentiable structure, and even a unique reductive structure if N is finite. We will also show that the existence of reductive structures for all conditional expectations $E \in \mathcal{E}(M)$ forces the algebra M to be finite.

4.1 Differentiable Structure.

Next we state some definitions and three classical Banach-Lie group theory's theorems that will be used afterwards. As a general reference about this subject, see, for example, [20] or [19].

Definition 4.1 Given a Lie-Banach group G (complex analytic, real analytic, or C^∞), we denote by $L(G)$ the Lie algebra of G, which will be always identified (as a complex or real Banach space) with the tangent space $T_1(G)$ of G at the identity. A subgroup H of G is called a **regular** subgroup if it is also a Lie-Banach group (of the same type) and if T_1H is closed and complemented in T_1G.

Theorem 4.2 *Let G a Lie group, $H \subseteq G$ a subgroup such that there exist open sets U, V with $0 \in U$, $1 \in V$ and a decomposition $T_1(G) = X \oplus Y$ (as a Banach space) satisfying*

1. $\exp : U \to V$ *is a diffeomorphism*

2. $H \cap V = \exp(X \cap U)$.

Then H *is a regular subgroup of* G *and* $T_1(H) = X$.

Theorem 4.3 *Let* G *be a Lie group,* $H \subseteq G$ *a regular subgroup. Then*

1. G/H *has a unique structure of differentiable manifold such that* $G \to G/H$ *is a submersion*

2. $G \to G/H$ *is a principal bundle with structure group* H

3. *The action* $G \times G/H \to G/H$ *is smooth.*

Theorem 4.4 *If* H *is a subgroup of a Lie group* G *and the connected component* H_1 *of* 1 *in* H *is a regular subgroup of* G*, then* H *is a regular subgroup of* G *if and only if* H_1 *is open in* H

In the following Proposition we construct a conditional expectation that will be essential in order to characterize the tangent space of $\mathcal{S}(E)$ (see also [9]).

Proposition 4.5 *Let* $N \subseteq M$ *be von Neumann algebras and* $E \in \mathcal{E}(M, N)$*. Fix a faithful normal state* φ *on* N*, and call* $\psi = \varphi \circ E$*. Then there exists a unique conditional expectation* $F \in \mathcal{E}(M, M_E)$ *such that* $EF = FE$ *and* $\psi \circ F = \psi$*.*

Proof. Denote by σ_t^ψ, $t \in I\!R$, the modular group of M induced by ψ. Since $\psi = \varphi \circ E = \psi \circ E$, we have that $\sigma_t^\psi \circ E = E \circ \sigma_t^\psi$ for all $t \in I\!R$ (see [9] or [28]). By direct computations we can deduce that $\sigma_t^\psi(M_E) = M_E$ for every $t \in I\!R$. Take $F \in \mathcal{E}(M, M_E)$ to be the unique expectation with $\psi \circ F = \psi$ obtained by Takesaki's Theorem on the existence of conditional expectations [28]. Since $E|_{M_E} \in \mathcal{E}(M_E, \mathcal{Z}(N))$, then $E \circ F \in \mathcal{E}(M, \mathcal{Z}(N))$ and $\psi \circ (E \circ F) = \psi$. When we represent M as usual in $L^2(M, \psi)$, the three conditional expectations $E, F, E \circ F$ give rise to three orthogonal projections e, f, g with $g = ef$. Since $g = g^*$, we have that $ef = fe$, so $EF = FE$. □

Using the expectation $F : M \to M_E$ from Proposition 4.5, we can define

$$\Delta = E + F - EF \in \mathcal{B}(M). \tag{16}$$

Δ is a projection, since E and F commute. Its image is the closed subspace $M_E + N$ of M, which can also be written as a direct sum:

$$\operatorname{Im} \Delta = (M_E \cap \ker E) \oplus N.$$

Proposition 4.6 *With the preceding notations, $\mathcal{Z}_E$ is a regular subgroup of G_M and $T_1\mathcal{Z}_E = (M_E \cap \ker E) \oplus N$.*

Proof. In order to use Theorem 4.2, we need a decomposition

$$T_1 G_M = X \oplus Y$$

with $X = M_E + N$, the natural candidate to be $T_1\mathcal{Z}_E$. This decomposition exists because as G_M is open in M, we have that $T_1G_M = M$ and so the projection Δ introduced in the preceding discussion gives the desired decomposition.

Note that the exponential map of the Banach-Lie group G_M coincides with the usual exponential map ($m \mapsto e^m$) under the identification of $L(G_M)$ with M. As exp is a local diffeomorphism we can fix an open set $0 \in U$ such that $\exp : U \to V = \exp(U)$ is a diffeomorphism. Let $0 \in U' \subseteq U$ be an open set and $x \in U' \cap X$; then $x = a + b$ with $a \in M_E$, $b \in N$, so (as a and b commute), we have $\exp(a+b) = \exp(a)\exp(b)$ with $\exp(a) \in G_{M_E}$ and $\exp(b) \in G_N$, thus showing that $\exp(U' \cap X) \subseteq \exp(U') \cap \mathcal{Z}_E$.

Let $0 < \delta < 1/2$ such that

$$B_M(1,\delta) = \{y \in M : \|y - 1\| < \delta\} \subseteq V.$$

Let $y \in B_M(1,\delta) \cap \mathcal{Z}_E$. Let $g \in M_E, h \in N$ with $y = gh$. Note that $F(h)$ is in $G_{\mathcal{Z}(N)}$. Indeed, since $h \in N$, we have that $F(h) \in \mathcal{Z}(N)$. To see that $F(h)$ is invertible note that

$$\|gF(h) - 1\| = \|F(gh - 1)\| \leq \|gh - 1\| < \delta < 1.$$

Now write $y = gh = (gF(h))(F(h)^{-1}h)$. Then

$$\|gF(h) - 1\| < \delta$$

as before and then, by Lemma 3.2,

$$\|F(h)^{-1}g^{-1} - 1\| < 2\delta.$$

Notice also that $\|gh - 1\| < \delta < 1$ implies $\|gh\| < 2$. Now, collecting estimates,

$$\begin{aligned} \|F(h)^{-1}h - 1\| &= \|F(h)^{-1}g^{-1}gh - 1\| \\ &\leq \|gh\|\|F(h)^{-1}g^{-1} - 1\| + \|gh - 1\| \\ &< 4\delta + \delta = 5\delta. \end{aligned}$$

Let $\varepsilon > 0$ small enough in order that $B_M(0, 2\varepsilon) \subseteq U$. Let δ small enough such that

$$\exp^{-1}(B_M(1, 5\delta)) \subseteq B_M(0, \varepsilon).$$

Call $V' = B_M(1,\delta) \subseteq V$, $U' = \exp^{-1}(V') \subseteq U$. Let $y \in V' \cap \mathcal{Z}_E$. Then $\exp^{-1}(y) \in U'$ and, since $gF(h)$ and $F(h)^{-1}h$ are in $B_M(1,5\delta)$, their preimages $a = \exp^{-1}(gF(h)) \in M_E$ and $b = \exp^{-1}(F(h)^{-1}h) \in N$ verify that $a+b \in U \cap X$. But $\exp(a+b) = y = \exp(\exp^{-1}(y))$ and exp is injective in U. Then $a+b = \exp^{-1}(y) \in U'$ and

$$\exp(U' \cap X) = V' \cap \mathcal{Z}_E$$

□

Corolary 4.7 *Let M be a von Neumann algebra and $E \in \mathcal{E}(M)$. Then, with the preceding notations, the isotropy group I_E is a regular subgroup of G_M.*

Proof. We already know that $\mathcal{Z}_E$ is a regular subgroup, that $\mathcal{Z}_E$ is the connected component of 1 in I_E and that it is open in I_E. So by Theorem 4.4, I_E is a regular subgroup. □

Theorem 4.8 *Let M be a von Neumann algebra and $E \in \mathcal{E}(M)$ a faithful normal conditional expectation. Then the similarity orbit $\mathcal{S}(E) \simeq G_M/I_E$, considered with the quotient topology of the norm topology of G_M, can be given a unique complex analytic manifold structure such that it is an homogeneous space (i.e. the map $\Pi_E : G_M \to \mathcal{S}(E)$ is a principal bundle with group structure I_E and $\Pi_E : G_M \to \mathcal{S}(E)$ is a submersion).*

Proof. Apply Corollary 4.7 and Theorem 4.3. □

Remark 4.9 The analogue of Theorem 4.8 is still true (but replacing complex analytic by real analytic) for the unitary orbit $\mathcal{U}(E) \simeq \mathcal{U}_M/\mathcal{N}_E$ under the action of the real analytic Banach-Lie group $\mathcal{U}_M$.

4.2 Reductive Structure.

Now we will start considering the conditions that will allow us to find a reductive structure in $\mathcal{S}(E)$ and to characterize it. First we recall the definition of Homogeneous Reductive Spaces (see also [23]):

Definition 4.10 *A Homogeneous Reductive Space (HRS) is a differentiable manifold $\mathcal{Q}$ and a smooth transitive action of a Banach-Lie group G on $\mathcal{Q}$, $L : G \times \mathcal{Q} \to \mathcal{Q}$ with:*

1. **Homogeneous Structure:** *for each $\rho \in \mathcal{Q}$ the map*

$$\begin{array}{rcl} \Pi_\rho : G & \to & \mathcal{Q} \\ g & \mapsto & L_g \rho \end{array}$$

is a principal bundle with structure group $I_\rho = \{g \in G : L_g\rho = \rho\}$ (called the isotropy group of ρ).

2. **Reductive Structure:** *for each $\rho \in \mathcal{Q}$ there exists a closed linear subspace H_ρ of the Lie algebra $L(G)$ of G such that $L(G) = H_\rho \oplus L(I_\rho)$ which is invariant under the natural action of I_ρ and such that the distribution $\rho \mapsto H_\rho$ is smooth.*

In order to give a HRS structure to the orbit $\mathcal{S}(E)$ under the action of G_M, we must find a decomposition

$$L(G_M) = L(I_E) \oplus \mathcal{K}_E,$$

such that the "horizontal" space $\mathcal{K}_E$ verifies

$$g(\mathcal{K}_E)g^{-1} = \mathcal{K}_E \quad \text{for all} \quad g \in I_E. \tag{17}$$

Recall that $L(G_M)$ can be identified with M, because G_M is open in M. Also, $L(I_E)$ can be regarded as $T_1 I_E$, and also as $T_1(I_E)_1$ (where $(I_E)_1$ is the connected component of I_E at 1). We have shown in Proposition 3.3 that the connected component if I_E at 1 is $\mathcal{Z}_E$, so $T_1 I_E = T_1 \mathcal{Z}_E = L(\mathcal{Z}_E)$, and by Proposition 4.6, $L(\mathcal{Z}_E) = M_E + N$. Therefore we already know (see equation (16)) that such a decomposition of M can be found. The problem which arises now is that we need a complement of $M_E + N$ verifying the equivariance property described in equation (17).

Lemma 4.11 *Let $B \subseteq A$ algebras, $P : A \to B$ a linear projection and $g \in G_A$ such that $g(\ker P)g^{-1} \subseteq \ker P$ and $gBg^{-1} = B$. Then $P(gxg^{-1}) = gP(x)g^{-1}$ for every $x \in A$.*

Proof. Straightforward. □

Lemma 4.12 *Let $N \subseteq M$ von Neumann algebras, $E \in \mathcal{E}(M, N)$ a faithful normal conditional expectation. Suppose that there exists a faithful normal* **tracial** *state φ of N. Let $\psi = \varphi \circ E$ and $F \in \mathcal{E}(M, M_E)$ as in Proposition 4.5. Then*

1. *The expectation F is unique in the sense that for any other faithful normal tracial state ρ in N, the expectation $F_\rho \in \mathcal{E}(M, M_E)$ induced by $\rho \circ E$ verifies $F_\rho = F$.*

2. $I_E \subseteq I_F = \{g \in G_M : gF(\cdot)g^{-1} = F(g \cdot g^{-1})\}$.

Proof. We shall first show the uniqueness of F. For every faithful normal tracial state ρ of N, the corresponding $F_\rho \in \mathcal{E}(M, M_E)$ given by Proposition 4.5 verifies that $F_\rho|_N \in \mathcal{E}(N, \mathcal{Z}(N))$ is the center valued trace of N, since $\rho \circ F_\rho|_N = \rho$ (see for example 8.3.10 of [17]). Then

$$\psi \circ F_\rho = \psi \circ E \circ F_\rho = \psi \circ F_\rho \circ E = \psi \circ F_\rho|_N \circ E = \psi \circ F|_N \circ E = \psi \circ F.$$

So $F_\rho = F$. Fix now $g \in I_E$. It is easy to see that $gM_E g^{-1} = M_E$. Taking the polar decomposition $g = |g^*|u$, we know by Proposition 3.1 and the proof of Theorem 3.5 that $u \in I_E \cap \mathcal{U}_M = \mathcal{N}_E$ and $|g^*| \in \mathcal{Z}_E$. Let us first see that $u \in I_F$. Indeed the expectation $F_u = L_u(F) = uF(u^* \cdot u)u^*$ verifies that $F_u \in \mathcal{E}(M, M_E)$, $F_u \circ E = E \circ F_u$ and $\varphi(u \cdot u^*) \circ E \circ F_u = \varphi(u \cdot u^*) \circ E$. That is, F_u is the expectation which corresponds by Proposition 4.5 to the trace $\varphi(u \cdot u^*)$ of N. By item 1 we can deduce that $F_u = F$, so $u \in I_F$. Therefore it suffices to show that $\mathcal{Z}_E \subseteq I_F$ and, since $\mathcal{Z}_E = G_{M_E} G_N$, to show that $G_N \subseteq I_F$. Let $g \in G_N$, $y \in M_E$ and $x \in \ker F$. Then, using that $\psi \circ F = \psi$, we get

$$\begin{aligned}
\psi(F(gxg^{-1})y) &= \psi(F(gxg^{-1}y)) = \psi(gxg^{-1}y) \\
&= \varphi(E(gxg^{-1}y)) = \varphi(gE(xg^{-1}yg)g^{-1}) \\
&= \varphi(E(xg^{-1}yg)) = \psi(F(xg^{-1}yg)) \\
&= \psi(F(x)g^{-1}yg) = 0.
\end{aligned}$$

Then $F(gxg^{-1}) = 0$, since $F(gxg^{-1}) \in M_E$ and ψ is faithful. Thus $g(\ker F)g^{-1} \subseteq \ker F$. By Lemma 4.11, we conclude that $I_E \subseteq I_F$, showing item 2 □

Proposition 4.13 *Let $N \subseteq M$ be von Neumann algebras, $E \in \mathcal{E}(M, N)$ a faithful normal conditional expectation and assume that N is finite. Then the similarity orbit $\mathcal{S}(E)$ has a unique HRS structure under the action of G_M.*

Proof. To find a reductive structure, we need to construct a decomposition $L(G_M) = L(I_E) \oplus \mathcal{K}_E$, where $\mathcal{K}_E$ is invariant by inner conjugation of elements of I_E. Fix a faithful normal tracial state φ of N and consider $F \in \mathcal{E}(M, M_E)$ induced by φ as in Proposition 4.5 and Lemma 4.12. By the discussion preceding Proposition 4.6, it is clear that the projection $\Delta = I - (I - E)(I - F)$ gives the desired decomposition, i.e. $\mathcal{K}_E = \ker \Delta$.

Now it remains to show that I_E leaves $\mathcal{K}_E$ invariant, and that the distribution $L_g E \mapsto g\mathcal{K}_E$ is smooth. The first assertion follows, since $\mathcal{K}_E = \ker F \cap \ker E$ and, by Lemma 4.12, $I_E \subseteq I_F$.

To see that the distribution is smooth, note that the projection onto $\mathcal{K}_E$ with kernel $M_E + N$ is $I - \Delta = D = (1-E)(1-F)$. By Lemma 4.12, the map

$$\eta : \mathcal{S}(E) \to \mathcal{B}(M) \quad \text{given by} \quad \eta(\Pi_E(g)) = L_g D = (1 - L_g E)(1 - L_g F), \quad g \in G_M$$

is well defined and gives the desidered decomposition for all $\Pi_E(g) \in \mathcal{S}(E)$. Consider the commutative diagram

$$\begin{array}{ccc} G_M & \xrightarrow{Ad} & Gl(\mathcal{B}(M)) \\ \Pi_E \downarrow & & \downarrow \Pi_D \\ \mathcal{S}(E) & \xrightarrow{\eta} & \mathcal{B}(M) \end{array}$$

where $\Pi_D(\alpha) = \alpha \circ D \circ \alpha^{-1}$, $\alpha \in Gl(\mathcal{B}(M))$. As we know that Π_E has analytic local cross sections by Theorem 4.8, the map η is clearly analytic.

The uniqueness follows from the fact that our selection of $\mathcal{K}_E$ (actually the expectation F) does not depend on the tracial state φ. Indeed, it is easy to see that for every faithful normal tracial state ρ of N, the corresponding $F_\rho \in \mathcal{E}(M, M_E)$ given by Proposition 4.5 verifies that $F_\rho|_N \in \mathcal{E}(N, \mathcal{Z}(N))$ is the center valued trace of N, since $\rho \circ F_\rho|_N = \rho$. Then $F_\rho = F$ □

Remark 4.14 Let $N \subseteq M$ be von Neumann algebras, $E \in \mathcal{E}(M, N)$ a faithful normal conditional expectation and assume that $M_E \subseteq N$ even though N was not necessarily finite. Then the assertion of Theorem 4.13 holds with the same proof. Indeed, in this case $\mathcal{Z}_E = G_N$ and one does not need a tracial state of N since $\Delta = E$. This fact was already shown in [21] under the slightly more restrictive hypothesis that $N' \cap M \subseteq N$.

4.15 Let M be an infinite von Neumann algebra. Then there exists a properly infinite projection $p \in \mathcal{Z}(M)$ such that pM is properly infinite and $(1-p)M$ is finite. Let τ be a faithful normal trace in $(1-p)M$. Since p is properly infinite, it can halved, i.e. there exists a projection $q \in M$ such that $q \sim p - q \sim p$, where $\sim$ denotes the von Neumann equivalence of projections. Using this projection q, we can identify pM with $qMq \otimes M_2(\mathbb{C})$. So we identify M with $(qMq) \otimes M_2(\mathbb{C})) \oplus (1-p)M$.

Let N be the subalgebra $(qMq \otimes 1) \oplus (1-p)\mathbb{C}$ of M. Consider the expectation $E \in \mathcal{E}(M, N)$ given by

$$E = (\mathrm{id} \otimes \mathrm{tr}_2) \oplus \tau.$$

In matrices this can be seen as

$$E\left(\begin{pmatrix} a & b & 0\\ c & d & 0\\ 0 & 0 & x\end{pmatrix}\right) = \begin{pmatrix} \frac{a+d}{2} & 0 & 0\\ 0 & \frac{a+d}{2} & 0\\ 0 & 0 & \tau(x)\end{pmatrix}.$$

Straightforward calculations show that

$$N'\cap M = \mathcal{Z}(qMq)\otimes M_2(\mathbb{C})\oplus(1-p)M$$

and that

$$M_E = N'\cap M.$$

If $\mathcal{S}(E)$ admits an Homogeneous Reductive Structure, then there exists a bounded linear projection $P : M \to N + M_E$ with $g(\ker P)g^{-1} = \ker P$ for all $g \in I_E$. Since $\mathcal{U}_N 1 I_E$, then

$$P(uxu^*) = uP(x)u^* \text{ for every } u \in \mathcal{U}_N$$

by Lemma 4.11. Note that, as $(N + M_E)^* = N + M_E$, P can be assumed to be $*$-linear. Indeed, if P is not $*$-linear, we can replace it by

$$P'(x) = \frac{1}{2}(P(x) + P(x^*)^*), x \in M.$$

This P' is also a projection onto $N + M_E$ and

$$P'(uxu^*) = uP'(x)u^* \quad \text{for every} \quad u \in \mathcal{U}_N.$$

Since we know that

$$N + M_E = \left\{\begin{pmatrix} n & z_2 & 0\\ z_3 & n+z_1 & 0\\ 0 & 0 & m\end{pmatrix} : n \in qMq, z_i \in \mathcal{Z}(qMq), m \in (1-p)M\right\}, \quad (18)$$

it is clear that the elements in coordinates 2 1 and 1 2 of the image of P belong to $\mathcal{Z}(qMq)$. Consider the linear map $T : qMq \to \mathcal{Z}(qMq)$ given by

$$T(n) = \frac{1}{2}\left(P\begin{pmatrix} 0 & n & 0\\ n & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_{21} + P\begin{pmatrix} 0 & n & 0\\ n & 0 & 0\\ 0 & 0 & 0\end{pmatrix}_{12}\right), \; n \in qMq$$

where $(\cdot)_{21}$ and $(\cdot)_{12}$ mean the 2 1 and the 1 2 coordinate of the matrix respectively. Now we show the properties of T that we will be of interest to us:

Proposition 4.16 *Let M be an infinite von Neumann algebra. Let p, q, N and $E \in \mathcal{E}(M,N)$ as in 4.15. Assume that the orbit $\mathcal{S}(E)$ admits an Homogeneous Reductive Structure. Consider the linear maps P and T as before. Then the following properties are satisfied:*

1. *$T : qMq \to \mathcal{Z}(qMq)$ is a $*$-linear mapping;*
2. *T is a projection onto $\mathcal{Z}(qMq)$;*
3. *if $u \in \mathcal{U}_{qMq}$, then $T(unu^*) = T(n)$ for every $n \in qMq$;*
4. *$T(xy) = T(yx)$ for every $x, y \in qMq$.*

Proof.

1. That the image of T is in $\mathcal{Z}(qMq)$ can be seen in equation (18). $*$-linearity is clear since we assume P to be $*$-linear.
2. If $s \in \mathcal{Z}(qMq)$, then the matrix

$$\begin{pmatrix} 0 & s & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is clearly in $N + M_E$, so the matrix is left invariant by P, and $T(s) = s$.

3. Let $u \in \mathcal{U}_{qMq}$ and consider

$$U = \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathcal{U}_N \subseteq I_E.$$

The basic property of P is that $P(UmU^*) = UP(m)U^*$ for every $m \in M$. But this clearly implies that $T(unu^*) = uT(n)u^* = T(n)$ for every $n \in qMq$.

4. Follows from 3 since the unitaries generate the whole algebra.

□

Theorem 4.17 *Let M be a von Neumann algebra. Then the following conditions are equivalent:*

1. *The similarity orbit $\mathcal{S}(E)$ of any expectation $E \in \mathcal{E}(M)$ can be given a HRS structure under the action of G_M.*

2. M is a finite von Neumann algebra.

Proof. Let p be the biggest projection in $\mathcal{Z}(M)$ such that pM is properly infinite, and q a subprojection of p that halves p, that is $q \sim p - q \sim p$. We shall use the notations of 4.15 and the conditional expectation $E \in \mathcal{E}(M)$ considered there. If condition 1. holds, using Proposition 4.16 and 4.15, we can construct a "tracial" bounded projection $T : qMq \to \mathcal{Z}(qMq)$. Since q is also properly infinite, there is a projection $r \in qMq$ such that $r \sim q - r \sim q$ in qMq. Using "traciality" of T,

$$T(q) = T(q - r) = T(q) - T(r) = T(q) - T(q) = 0. \tag{19}$$

Recall from proposition 4.16 that $T(q) = q$, so by equation (19) we have that $q = 0$, and this implies that $p = 0$. So M is a finite von Neumann algebra.

Conversely, suppose that M is finite and $E \in \mathcal{E}(M)$. Then $N = E(M)$ is a finite von Neumann algebra and we can apply Proposition 4.13. □

Remark 4.18 Let $N \subseteq M$ von Neumann algebras and $E \in \mathcal{E}(M, N)$ such that $\mathcal{S}(E)$ has a structure of HRS. We shall describe explicitly the geometrical invariants of $\mathcal{S}(E)$. First we compute the tangent map at 1 of the fibration $\Pi_E : G_M \to \mathcal{S}(E)$. For simplicity we shall consider $\mathcal{S}(E) \subseteq \mathcal{B}(M)$, in spite of the fact that the topology considered in $\mathcal{S}(E)$ is not in general that induced by $\mathcal{B}(M)$. In this sense, for $x \in M$,

$$(\mathrm{T}\ \Pi_E)_1(x) = [x, E(\cdot)] - E([x, \cdot]),$$

where $[x, y] = xy - yx$, for $x, y \in M$. Indeed, let $x \in M$ and consider the curve $\alpha(t) = e^{tx}$. Note that $\alpha(0) = 1$ and $\dot\alpha(0) = x$. Then

$$\begin{aligned}
(\mathrm{T}\ \Pi_E)_1(x) &= \tfrac{d}{dt}(\Pi_E(e^{tx}))|_{t=0} \\
&= \tfrac{d}{dt}\left(\mathrm{Ad}(e^{tx}) \circ E \circ \mathrm{Ad}(e^{-tx})\right)|_{t=0} \\
&= \left((\mathrm{Ad}(e^{tx}))' \circ E \circ \mathrm{Ad}(e^{-tx}) + \mathrm{Ad}(e^{tx}) \circ E \circ (\mathrm{Ad}(e^{-tx}))'\right)|_{t=0} \\
&= \left((\mathrm{Ad}(e^{tx}))([x, E \circ \mathrm{Ad}(e^{-tx})]) + \mathrm{Ad}(e^{tx}) \circ E \circ (\mathrm{Ad}(e^{-tx})([\cdot, x])\right)|_{t=0} \\
&= [x, E(\cdot)] - E([x, \cdot]).
\end{aligned}$$

An interesting computation using this formula shows, as it must be, that

$$\ker(\mathrm{T}\ \Pi_E)_1 = M_E + N = L(I_E).$$

On the other hand, if $\mathcal{K}_E = \ker \Delta$ is the horizontal space at E of $\mathcal{S}(E)$, then

$$(\mathrm{T}\ \Pi_E)_1|_{\mathcal{K}_E} : \mathcal{K}_E \to T(\mathcal{S}(E))_E$$

is an isomorphism. It is usual consider its inverse $K_E : T(\mathcal{S}(E))_E \to \mathcal{K}_E$ in order to identify tangent vectors with elements of M (see, for instance, [23]). With this convention we shall describe the torsion and curvature tensors, T and R, respectively. Let V, W and $Z \in T(\mathcal{S}(E))_E$. Then

1. $T(V, W) = (\mathrm{T}\ \Pi_E)_1([K_E(V), K_E(W)])$.
2. $R(V, W)Z = (\mathrm{T}\ \Pi_E)_1([K_E(Z), \Delta([K_E(V), K_E(W)])])$.
3. The unique geodesic γ at E such that $\dot\gamma(0) = V$ is given by
$$\gamma(t) = L_{e^{tK_E(V)}} E.$$
4. The exponential map of $\mathcal{S}(E)$ is given by
$$\exp_E(X) = L_{e^{K_E(X)}} E \quad \text{for} \quad X \in T(\mathcal{S}(E))_E.$$

References

1. E. Andruchow, G. Corach and D. Stojanoff; A geometric characterization of nuclearity and injectivity, J. Funct. Anal. **133** (1995), 474-494.

2. E. Andruchow, G. Corach and D. Stojanoff, Geometry of the sphere of a Hilbert module, Math. Proc. of the Cambridge Math. Soc., **127** (1999) 295-316.

3. E. Andruchow, A. R. Larotonda, L. Recht and D. Stojanoff; Infinite dimensional homogeneous reductive spaces and finite index conditional expectations, Illinois J. of Math. **41** (1997), 54-76.

4. E. Andruchow, D. Stojanoff, "Geometry of conditional expectations and finite index", Int. Journal of Math. **5** (1994), 169-178.

5. E. Andruchow and A. Varela, Homotopy of State orbits, preprint (1998).

6. H.Araki, M.Smith and L. Smith, On the homotopical significance of the type of von Neumann algebra factors, Commun. Math. Phys. **22** (1971),71-88.

7. M. Argerami, D. Stojanoff, "The Weyl Group and the Normalizer of a Conditional Expectation", Int. Eq. and Op. Theory, 34 (1999), 165-186.

8. M. Argerami, D. Stojanoff, “Some examples of the Weyl Group of Conditional Expectations”, preprint.

9. F. Combes, C. Delaroche, “Groupe modulaire d’une esperance conditionnelle dans une algebre de von Neumann”, Bull. Soc. Math. France **103** (1975), 385-426.

10. A. Connes, “Une classification des facteurs de type III”, Ann. Ec. Norm. Sup. Paris **6** (1973), 133-252.

11. G. Corach, H. Porta and L. Recht, Differential geometry of systems of projections in Banach algebras, Pacific J. Math. **143** (1990), no. 2, 209-228.

12. G. Corach, H. Porta and L. Recht, The geometry of spaces of projections in C*-algebras, Adv. in Math. **101** (1993), 59-77.

13. M. J. Greenberg, “Lectures on Algebraic Topology”, W. A. Benjamin, inc., New York, 1967.

14. J.-F. Havet, “Esperance conditionelle minimale”, J. Oper. Th. **24** (1990), 33-55.

15. D. E. Handelman, K_0 of von Neumann algebras and AFC*-algebras, Quart. J. Math. Oxford (2) **29** (1978), 429-441.

16. V.F.R. Jones, “Index for subfactors”, Invent. Math., **72** (1983), 1-25.

17. R. Kadison, John R. Ringrose, “Fundamentals of the Theory of Operator Algebras”, II, Academic Press, New York 1986.

18. H. Kosaki, “Characterization of Crossed Product (Properly Infinite Case)”, Pacific J. of Math., **137** (1989), 159-167.

19. S. Lang, *Differentiable Manifolds*, Addison–Wesley, Reading, Mass. 1972

20. A. R. Larotonda, *Notas sobre variedades diferenciales, Notas de Geometría y Topología 1*, INMABB-CONICET, Universidad Nacional del Sur, Bahia Blanca (1980)

21. A. Larotonda, L. Recht, “The orbit of a conditional expectation as a reductive homogeneous space”, Monteiro, Luiz (ed.), Homage to Dr. Rodolfo A. Ricabarra. Bahia Blanca: Univ. Nacional del Sur, 61-73 (1995).

22. R. Longo, “Index of subfactors and statistics of quantum fields I and II”, Comm. Math. Phys. **126** (1989), 217-247, **130** (1990), 285-309.

23. L. Mata Lorenzo, L. Recht, Infinite dimensional homogeneous reductive spaces, Reporte 91-11, U.S.B., (1991).

24. M. Pimsner, S. Popa, “Entropy and index for subfactors”, Ann. scient. Ec. Norm. Sup., 4ta serie, **19** (1986), 57-106.

25. S. Popa, “Classification of subfactors and their endomorphisms”, AMS series CBMS **86** (1995).

26. H. Porta and L. Recht; Minimality of geodesics in Grassmann manifolds, Proc. Amer. Math. Soc., 100, (1987), 464-466.

27. H. Schröder, On the homotopy type of the regular group of a W*-algebra, Math. Ann. **267** (1984), 271-277.

28. M. Takesaki, Conditional Expectations in von Neumann Algebras, J. Funct. Anal. **9** (1972) 306-321.

Martín Argerami and Demetrio Stojanoff
Dpto. de Matemática
Fac. Cs. Exactas – UNLP
CC 172 (1900) La Plata
Argentina
email: martin@mate.unlp.edu.ar & demetrio@mate.dm.uba.ar

AMS Classification Numbers: 46L10 and 46L99.