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Young's inequality in trace-class operators

Martín Argerami · Douglas R. Farenick

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Abstract. If a and b are trace-class operators, and if u is a partial isometry, then $\|u \mid ab^*|u^*\|_1 \leq \|\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\|_1$, where $\|\cdot\|_1$ denotes the norm in the trace class. The present paper characterises the cases of equality in this Young inequality, and the characterisation is examined in the context of both the operator and the Hilbert–Schmidt forms of Young's inequality.

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1. Introduction

Building on the work by T. Ando [A], it was shown in [EFZ] that for each pair of compact operators a and b, and for each $p, q \in \mathbb{R}^+$ for which $1/p + 1/q = 1$, there exists a partial isometry u such that $u^*u = 1 - [\ker |ab^*|]$ (where $[\ker |ab^*|]$ denotes the projection onto $\ker |ab^*|$) and

$$u|ab^*|u^* \leq \frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q. \tag{1.1}$$

The case $p = q = 2$ is the arithmetic-geometric mean inequality, which was first uncovered in [BK]. In finite dimensions, the operator u in (1.1) can always be required to be a unitary.

The problem of characterising the cases of equality in the operator Young inequality (1.1) was settled by O. Hirzallah and F. Kittaneh in [HK] for Hilbert–Schmidt operators. A very surprising feature of Hirzallah and Kittaneh's proof is that the analysis of the cases of equality in the operator inequality (1.1) occurs through a much weaker assumption. Specifically, the fact that the operators $u|ab^*|u^*$ and $\frac{1}{p}|a|^p + \frac{1}{p}|b|^q$ are equal has no role in the proof; rather, one need only know that the Hilbert–Schmidt norms of the operators $u|ab^*|u^*$ and $\frac{1}{p}|a|^p + \frac{1}{p}|b|^q$ are equal.

M. Argerami
Departamento de Matemática, Facultad de Ciencias Exactas, Universidad Nacional de La Plata, CC 172 (1900) La Plata, Argentina (e-mail: `martin@mate.unlp.edu.ar`)

D. R. Farenick
Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan S4S 0A2, Canada (e-mail: `farenick@math.uregina.ca`)

In this paper we prove that for trace-class operators the Hilbert–Schmidt condition can replaced by equality in trace norm (Theorems 2.1 and 2.2). Moreover, in Theorem 4.1 we show that

$$\| u|ab^*|u^* \|_1 = \left\| \frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q \right\|_1$$

if and only if

$$\| u|ab^*|u^* \|_2 = \left\| \frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q \right\|_2 ,$$

but neither of these equalities imply operator equality in (1.1).

Motivated by the operator Young inequality (1.1), our results incorporate the use of partial isometries as an added feature. Our computations rely on a careful analysis of the eigenvalues of the operators, and also on some convexity arguments about perturbations by unitaries.

We now introduce some notation and conventions. Suppose that $\mathcal{H}$ is a complex, separable Hilbert space and that $B(\mathcal{H})$ is the algebra of bounded linear operators acting on $\mathcal{H}$. For $x \in B(\mathcal{H})$, we denote by $R[x]$ the projection of $\mathcal{H}$ onto the closed range $\overline{\text{ran}\, x}$ of x. For any operator z, $|z|$ shall denote $(z^*z)^{1/2}$, the positive square root of z^*z. If $z \in B(\mathcal{H})$ is compact, then the nonnegative real number $s_k(z)$, for every $k \in \mathbb{Z}^+$, denotes the k-th singular value of z, namely

$$s_k(z) = \lambda_k(|z|),$$

where

$$\lambda_k(|z|) = \min\Big\{ \max\{\langle |z|\xi, \xi\rangle : \xi \in M^\perp,\ \|\xi\| = 1\} : M \subset H,\ \dim M = k-1 \Big\}.$$

Trace-class and Hilbert–Schmidt operators are defined via the sequence of singular values. An operator x is of trace class if $\{s_k(x)\}_{k\in\mathbb{Z}^+} \in \ell^1(\mathbb{Z}^+)$, and x is a Hilbert–Schmidt operator if $\{s_k(x)\}_{k\in\mathbb{Z}^+} \in \ell^2(\mathbb{Z}^+)$. Thus, the trace norm $\|\cdot\|_1$ and Hilbert–Schmidt norm $\|\cdot\|_2$ on the ideals of trace-class operators and Hilbert–Schmidt operators, respectively, are:

$$\|x\|_1 = \sum_{k=1}^{\infty} s_k(x) \tag{1.2}$$

and

$$\|x\|_2 = \left(\sum_{k=1}^{\infty} s_k(x)^2 \right)^{1/2} . \tag{1.3}$$

The trace $\operatorname{tr}(x)$ of a trace-class operator x is defined to be

$$\operatorname{tr}\,(x) = \sum_{k=1}^{\infty} \langle x\phi_k, \phi_k\rangle , \tag{1.4}$$

where $\{\phi_k\}_{k\in\mathbb{Z}^+}$ is any orthonormal basis of $\mathcal{H}$. It is well known that the trace is independent of the choice of orthonormal basis and that $\mathrm{tr}\,(xg) = \mathrm{tr}\,(gx)$ if $x \in B(\mathcal{H})$ is of trace class and $g \in B(\mathcal{H})$ is arbitrary.

If z is a positive compact operator, then the singular values of z are simply the eigenvalues of z and the list $\{\lambda_k(z)\}_{k\in\mathbb{Z}^+}$ accounts for all the nonzero eigenvalues of z (of which there may be finitely or infinitely many). Hence, equations (1.2) and (1.4) become, for positive trace-class operators z,

$$\|z\|_1 = \mathrm{tr}\; z = \sum_{k=1}^{\infty} \lambda_k(z)\,.$$

Finally, we will also make use of some basic facts from the theory of majorisation, for which our main reference is the book of Bhatia [Ba].

2. Equality in Young's inequality

We begin by mentioning the trivial fact that if $p, q \in \mathbb{R}^+$ and $1/p + 1/q = 1$, then $p > 1, q > 1$. Thus, if $a, b \in B(\mathcal{H})$ are positive trace-class operators, then both a^p and b^q are positive trace-class operators.

The following theorem is the main result of the present paper.

Theorem 2.1. *Let $a, b \in B(\mathcal{H})$ be positive trace-class operators, and suppose that $u \in B(\mathcal{H})$ is a partial isometry. Assume that $p, q \in \mathbb{R}^+$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$tr\,(u|ab|u^*) = tr\left(\frac{1}{p}\,a^p + \frac{1}{q}\,b^q\right)$$

if and only if

$$b^q = a^p, \quad \text{and} \quad R[u^*] \geq R[b].$$

Proof. Assume first that $b^q = a^p$, and $R[u^*] \geq R[b]$. This last condition can be stated as $u^*ub = b$. Note that

$$ab = b^{q/p+1} = b^q.$$

Then,

$$\mathrm{tr}\,(u|ab|u^*) = \mathrm{tr}\,(u|b^q|u^*) = \mathrm{tr}\,(ub^q u^*)$$

$$= \mathrm{tr}\,(u^*ub^q) = \mathrm{tr}\,(b^q)$$

$$= \mathrm{tr}\left(\frac{1}{p}\,a^p + \frac{1}{q}\,b^q\right),$$

which proves the "if" part.

Now we assume that a, b are positive trace-class operators satisfying

$$\operatorname{tr}(u|ab|u^*) = \frac{1}{p}\operatorname{tr}(a^p) + \frac{1}{q}\operatorname{tr}(b^q), \tag{2.1}$$

for some partial isometry u. In order to prove that $a^p = b^q$, we can assume without loss of generality that $u = 1$. Indeed, since u is a partial isometry we have that both u^*u and uu^* are projections, so

$$\begin{aligned}\frac{1}{p}\operatorname{tr}(a^p) + \frac{1}{q}\operatorname{tr}(b^q) &= \operatorname{tr}(u|ab|u^*) = \operatorname{tr}(|ab|^{1/2}u^*u|ab|^{1/2}) \\ &\leq \operatorname{tr}(|ab|) \leq \frac{1}{p}\operatorname{tr}(a^p) + \frac{1}{q}\operatorname{tr}(b^q)\,,\end{aligned} \tag{2.2}$$

the last inequality being derived from Proposition 3.3 of [EFZ] (Young's inequality in eigenvalues). Then, equations (2.1) and (2.2) imply

$$\operatorname{tr}(|ab|) = \frac{1}{p}\operatorname{tr}(a^p) + \frac{1}{q}\operatorname{tr}(b^q). \tag{2.3}$$

Equation (2.3) leads to the following inequalities (noted in [Z]):

$$\begin{aligned}\frac{1}{p}\sum_{j=1}^{\infty}\lambda_j(a)^p + \frac{1}{q}\sum_{j=1}^{\infty}\lambda_j(b)^q &= \frac{1}{p}\sum_{j=1}^{\infty}\lambda_j(a^p) + \frac{1}{q}\sum_{j=1}^{\infty}\lambda_j(b^q) \\ \text{(using (2.3))} &= \sum_{j=1}^{\infty}\lambda_j(|ab|) = \sum_{j=1}^{\infty}s_j(|ab|) \\ &= \sum_{j=1}^{\infty}s_j(ab) \\ &\leq \sum_{j=1}^{\infty}s_j(a)s_j(b) = \sum_{j=1}^{\infty}\lambda_j(a)\lambda_j(b) \\ &\leq \sum_{j=1}^{\infty}\left(\frac{1}{p}\lambda_j(a)^p + \frac{1}{q}\lambda_j(b)^q\right) \\ &= \frac{1}{p}\sum_{j=1}^{\infty}\lambda_j(a)^p + \frac{1}{q}\sum_{j=1}^{\infty}\lambda_j(b)^q\,.\end{aligned}$$

In the above we have used [GK, page 63] and the scalar Young inequality for the first and second inequalities respectively. The inequalities above imply that

$$\sum_{j=1}^{\infty} \lambda_j(a)\lambda_j(b) = \frac{1}{p} \sum_{j=1}^{\infty} \lambda_j(a)^p + \frac{1}{q} \sum_{j=1}^{\infty} \lambda_j(b)^q.$$

Moreover, we know for each j that $\lambda_j(a)\lambda_j(b) \le \frac{1}{p}\,\lambda_j(a)^p + \frac{1}{q}\,\lambda_j(b)^q$, and so the equality obtained above implies that $\lambda_j(a)\lambda_j(b) = \frac{1}{p}\,\lambda_j(a)^p + \frac{1}{q}\,\lambda_j(b)^q$ for every $j \in \mathbb{Z}^+$. Since these are cases of equality in the scalar Young inequality, we have that $\lambda_j(a)^p = \lambda_j(b)^q$ for every $j \in \mathbb{Z}^+$.

Now we express $\mathcal{H}$ as two direct sums:

$$\mathcal{H} = \overline{\operatorname{ran} a} \oplus \ker a = \overline{\operatorname{ran} b} \oplus \ker b.$$

Since the sequences $\{\lambda_k(a)\}_{k\in\mathbb{Z}^+}$ and $\{\lambda_k(b)\}_{k\in\mathbb{Z}^+}$ capture all the nonzero eigenvalues of a and b, with multiplicities counted, we deduce that the Hilbert spaces $\overline{\operatorname{ran} a}$ and $\overline{\operatorname{ran} b}$ are isomorphic. Let w_0 be a unitary implementing this isomorphism; it can be viewed as a partial isometry in $B(\mathcal{H})$ with initial space $\overline{\operatorname{ran} a}$ and final space $\overline{\operatorname{ran} b}$. The computations we have done to this stage do not give us any information about the relation between $\ker a$ and $\ker b$; in particular, there is no reason why $\ker a$ and $\ker b$ should be isomorphic. To address this difficulty, we consider new operators $\tilde{a}$ and $\tilde{b}$ on $\mathcal{H} \oplus \mathcal{H}$, namely $\tilde{a} = a \oplus 0$ and $\tilde{b} = b \oplus 0$. Notice that, if we consider $\mathcal{H}$ embedded as the "first coordinate" of $\mathcal{H} \oplus \mathcal{H}$, we have that $\operatorname{ran} \tilde{a} = \operatorname{ran} a$ and $\operatorname{ran} \tilde{b} = \operatorname{ran} b$. The benefit of this situation is that we now can say that $\ker \tilde{a}$ and $\ker \tilde{b}$ have the same (infinite) dimension, and therefore the partial isometry w_0 extends to a unitary w acting on $\mathcal{H} \oplus \mathcal{H}$ such that $\tilde{b}^q = w\tilde{a}^p w^*$.

For these extended operators we still have equality (2.3), which becomes

$$\operatorname{tr}(|\tilde{a}w\tilde{a}^{p/q}w^*|) = \operatorname{tr}(\tilde{a}^p) \tag{2.4}$$

when we take into account that now w is a unitary. Our aim now is to prove that the unitary w commutes with $\tilde{a}$. To achieve that, it is enough, by functional calculus, to show that w commutes with $\tilde{a}^p$. So in our formulas we can replace $\tilde{a}$ by $\tilde{a}^{1/p}$ without loss of generality: we will prove then that w commutes with $\tilde{a}^p$, which is equivalent to w commuting with $\tilde{a}$. Therefore, the equality that we now analyse is

$$\operatorname{tr}(|\tilde{a}^{1/p}w\tilde{a}^{1/q}w^*|) = \operatorname{tr}(\tilde{a}), \tag{2.5}$$

where we have renamed $\tilde{a}^p$ as $\tilde{a}$.

Equation (2.5) can also be written as $\mathrm{tr}\,(|\tilde{a}^{1/q} w_1 \tilde{a}^{1/p} w_1^*|) = \mathrm{tr}\,(\tilde{a})$, where w_1 is the unitary $w_1 = w^*$. To see why, note that if $y = \tilde{a}^{1/p} w \tilde{a}^{1/q}$, then

$$\mathrm{tr}\,(|\tilde{a}^{1/p} w \tilde{a}^{1/q} w^*|) = \mathrm{tr}\,((w y^* y w^*)^{1/2}) = \mathrm{tr}\,(w (y^* y)^{1/2} w^*) = \mathrm{tr}\,((y^* y)^{1/2}),$$

the middle equality being valid because w is unitary. Because $\ker(y^*y - \lambda 1)$ and $\ker(yy^* - \lambda 1)$ have the same (finite) dimension for all nonzero λ, it follows that $\mathrm{tr}\,((y^*y)^{1/2}) = \mathrm{tr}\,((yy^*)^{1/2}) = \mathrm{tr}\,((w^*yy^*w)^{1/2}) = \mathrm{tr}\,(|y^*w|) = \mathrm{tr}\,(|\tilde{a}^{1/q} w_1 \tilde{a}^{1/p} w_1^*|)$.

Therefore, we assume henceforth that $p \geq 2$, for otherwise we could simply exchange q for p and w_1 for w before continuing with the arguments that now follow.

The space $\mathcal{H} \oplus \mathcal{H}$ has an orthonormal basis $\{\phi_k\}_{k \in \mathbb{Z}^+}$ of eigenvectors of $\tilde{a}$ (infinitely many of them belonging to $\ker \tilde{a}$). Let $\{d_k\}_{k \in \mathbb{Z}^+} \subset \mathbb{R}_0^+$ be such that $\tilde{a}\phi_k = d_k \phi_k$ for every $k \in \mathbb{Z}^+$, and set $w_{ij} = \langle w\phi_j, \phi_i \rangle$, for all $i, j \in \mathbb{Z}^+$. Then, from $w^*w = ww^* = 1$,

$$\sum_{i=1}^{\infty} |w_{ij}|^2 = \sum_{j=1}^{\infty} |w_{ij}|^2 = 1 \tag{2.6}$$

and, for every j,

$$w^*\phi_j = \sum_{k=1}^{\infty} \langle w^*\phi_j, \phi_k \rangle \phi_k = \sum_{k=1}^{\infty} \overline{w}_{jk} \phi_k.$$

Using the equations above, the terms in equation (2.5) become

$$\mathrm{tr}\,(\tilde{a}) = \sum_{j=1}^{\infty} \langle \tilde{a}\phi_j, \phi_j \rangle = \sum_{j=1}^{\infty} d_j, \tag{2.7}$$

and

$$\begin{aligned} \mathrm{tr}\,(|\tilde{a}^{1/p} w \tilde{a}^{1/q} w^*|) &= \mathrm{tr}\,((w \tilde{a}^{1/q} w^* \tilde{a}^{2/p} w \tilde{a}^{1/q} w^*)^{1/2}) \\ &= \mathrm{tr}\,(w (\tilde{a}^{1/q} w^* \tilde{a}^{2/p} w \tilde{a}^{1/q})^{1/2} w^*) \\ &= \mathrm{tr}\,((\tilde{a}^{1/q} w^* \tilde{a}^{2/p} w \tilde{a}^{1/q})^{1/2}). \end{aligned} \tag{2.8}$$

If $\xi \in \mathcal{H}$ is a unit vector and $x \in B(\mathcal{H})^+$, then

$$\langle x^{1/2}\xi, \xi \rangle \leq \|x^{1/2}\xi\| \, \|\xi\| = \langle x\xi, \xi \rangle^{1/2}. \tag{2.9}$$

Thus, using equations (2.7), (2.8), and (2.9), we obtain from (2.5)

$$\sum_{j=1}^{\infty} d_j = \operatorname{tr}((\tilde{a}^{1/q} w \tilde{a}^{2/p} w^* \tilde{a}^{1/q})^{1/2}) = \sum_{j=1}^{\infty} \langle (\tilde{a}^{1/q} w \tilde{a}^{2/p} w^* \tilde{a}^{1/q})^{1/2} \phi_j, \phi_j \rangle$$

$$\leq \sum_{j=1}^{\infty} \langle (\tilde{a}^{1/q} w \tilde{a}^{2/p} w^* \tilde{a}^{1/q}) \phi_j, \phi_j \rangle^{1/2}$$

$$= \sum_{j=1}^{\infty} \left(d_j^{2/q} \langle w \tilde{a}^{2/p} w^* \phi_j, \phi_j \rangle \right)^{1/2}$$

$$= \sum_{j=1}^{\infty} \left(d_j^{2/q} \sum_{k,h=1}^{\infty} \overline{w}_{jk} w_{jh} \langle \tilde{a}^{2/p} \phi_k, \phi_h \rangle \right)^{1/2}$$

$$= \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} d_j^{2/q} d_k^{2/p} |w_{jk}|^2 \right)^{1/2} = \sum_{j=1}^{\infty} d_j^{1/q} \left(\sum_{k=1}^{\infty} d_k^{2/p} |w_{jk}|^2 \right)^{1/2}$$

$$\leq \left(\sum_{j=1}^{\infty} d_j \right)^{1/q} \left[\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} d_k^{2/p} |w_{jk}|^2 \right)^{p/2} \right]^{1/p},$$

where we have used Hölder's Inequality in the last step.

Consider now the vectors

$$x = (d_j)_j, \quad y = \left(\sum_{k=1}^{\infty} d_k^{2/p} |w_{jk}|^2 \right)_j.$$

Note that from equations (2.6), the matrix $A = \{|w_{jk}|^2\}_{jk}$ is doubly stochastic, so $y = A\, x^{2/p}$ is majorised by $x^{2/p}$. If we consider the vectors $x^{(n)} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow})$, $y^{(n)} = (y_1^{\downarrow}, \ldots, y_n^{\downarrow})$, where as usual the arrow means that we are considering the entries of x and y in decreasing order, then $y^{(n)}$ is weakly majorised by $(x^{(n)})^{2/p}$. Therefore, $f(y^{(n)})$ is weakly majorised by $f(x^{(n)})$, for every convex monotone function f (see Corollary II.3.4 of [Ba]), and in particular for the function $f(x) = x^{p/2}$ (recall that $p \geq 2$). Thus,

$$\operatorname{tr}(y^{(n)\,p/2}) \leq \operatorname{tr}(x^{(n)}) \text{ for every } n.$$

Hence,

$$\operatorname{tr}(y^{p/2}) \leq \operatorname{tr}(x);$$

that is,

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} d_k^{2/p}|w_{jk}|^2\right)^{p/2} \leq \sum_{j=1}^{\infty} d_j.$$

Taking into account this inequality and the previous computations, we get

$$\begin{aligned}
\sum_{j=1}^{\infty} d_j &\leq \sum_{j=1}^{\infty} d_j^{1/q}\left(\sum_{k=1}^{\infty} d_k^{2/p}|w_{jk}|^2\right)^{1/2} \\
&\leq \left(\sum_{j=1}^{\infty} d_j\right)^{1/q}\left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} d_k^{2/p}|w_{jk}|^2\right)^{p/2}\right]^{1/p} \\
&\leq \left(\sum_{j=1}^{\infty} d_j\right)^{1/q}\left(\sum_{j=1}^{\infty} d_j\right)^{1/p} \\
&= \sum_{j=1}^{\infty} d_j.
\end{aligned}$$

So in particular

$$\begin{aligned}
&\sum_{j=1}^{\infty} d_j^{1/q}\left(\sum_{k=1}^{\infty} d_k^{2/p}|w_{jk}|^2\right)^{1/2} \\
&\qquad = \left(\sum_{j=1}^{\infty} d_j\right)^{1/q}\left[\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty} d_k^{2/p}|w_{jk}|^2\right)^{p/2}\right]^{1/p}
\end{aligned}$$

Since this is equality in the Hölder inequality, we conclude that, for every j,

$$d_j = \left(\sum_{k=1}^{\infty} d_k^{2/p}|w_{jk}|^2\right)^{p/2},$$

which we may write as

$$d_j^{2/p} = \sum_{k=1}^{\infty} d_k^{2/p}|w_{jk}|^2 . \tag{2.10}$$

The sequence $\{d_k\}_k$ consists of eigenvalues of $\tilde{a}$, repeated according to multiplicity. Every nonzero eigenvalue of a (and hence of $\tilde{a}$) is repeated at most finitely many times, since a is compact; the eigenvalue zero appears infinitely many times.

Now assume that λ_1 is the largest positive eigenvalue of a, and that $m_1 = \dim\ker(a - \lambda_1 1)$. Then there exists a permutation τ of $\mathbb{Z}^+$ for which $d_{\tau(k)} = \lambda_1$ for $k = 1, \ldots, m_1$. Use this permutation to rewrite (2.10) as

$$d_{\tau(j)}^{2/p} = \sum_{k=1}^{\infty} d_{\tau(k)}^{2/p} |w_{\tau(j)\tau(k)}|^2, \qquad \text{for all } j \in \mathbb{Z}^+,$$

and to reorder the orthonormal basis of eigenvectors as $\{\phi_{\tau(k)}\}_k$. Thus, without loss of generality, we may assume that the sequence $\{d_k\}_k$ is already ordered in such a way that

$$\lambda_1 = d_1 = d_2 = \cdots = d_{m_1} > d_k \qquad \text{for every } k \geq m_1 + 1\,.$$

Therefore, we still work with equation (2.10). For every $j = 1, \ldots, m_1$, we have that $d_j^{2/p}$ is an extremal point of the interval $[0, \lambda_1^{2/p}]$. Because equation (2.10) represents, for $1 \leq j \leq m_1$, $\lambda_1^{2/p} = d_j^{2/p}$ as a convex combination of points from the sequence $\{d_k^{2/p}\}_k$, equality in (2.10) can hold for each $1 \leq j \leq m_1$ only if

$$|w_{jk}|^2 = 0 \qquad \text{for all } k \geq m_1 + 1, \quad 1 \leq j \leq m_1. \tag{2.11}$$

Thus, the matrix representation for the unitary w with respect to the orthonormal basis $\{\phi_k\}_k$ has the form

$$w = \begin{bmatrix} w_1 & 0 \\ x & w_2 \end{bmatrix},$$

where $w_1 \in M_{m_1}(\mathbb{C})$ is unitary. The condition $ww^* = 1$ implies that $w_1 x^* = 0$, and so $x^* = w_1^*(w_1 x^*) = 0$, whence $x = 0$. Therefore,

$$w_{st} = 0, \qquad \text{for every } 1 \leq t \leq m_1, \quad s \geq m_1 + 1\,, \tag{2.12}$$

and so $\ker(\tilde{a} - \lambda_1 1) = \mathrm{Span}\{\phi_1, \ldots, \phi_{m_1}\}$ is invariant under w and w^*. Thus $wp_1 = p_1 w$, where p_1 is the projection of $\mathcal{H} \oplus \mathcal{H}$ onto $\ker(\tilde{a} - \lambda_1 1)$.

Now consider the next largest positive eigenvalue λ_2 of $\tilde{a}$. Let $m_2 = \dim\ker(\tilde{a} - \lambda_2 1)$; then there exists a reordering of the sequence $\{d_k\}_k$ so that $\lambda_1 = d_k$ for $k = 1, \ldots, m_1$, and $\lambda_2 = d_k$ for $k = m_1 + 1, \ldots, m_1 + m_2$, and $\lambda_2 > d_j$ if $j \geq m_1 + m_2 + 1$. In this reordering, equation (2.10) becomes

$$\sum_{k=1}^{m_1} d_k^{2/p} |w_{jk}|^2 + \sum_{k=m_1+1}^{\infty} d_k^{2/p} |w_{jk}|^2 = d_j^{2/p}\,,$$

which by conditions (2.11) and (2.12) specialises to

$$\sum_{k=m_1+1}^{\infty} d_k^{2/p} |w_{jk}|^2 = d_j^{2/p}, \qquad \text{for all } m_1 + 1 \leq j \leq m_1 + m_2\,. \tag{2.13}$$

Now, because $d_{m_1+1}^{2/p}, \ldots, d_{m_1+m_2}^{2/p} > d_t^{2/p}$ for all $t \geq m_1 + m_2 + 1$, the arguments that were used above prove that equation (2.13) is possible only if $|w_{jk}| = 0$ for all $k \geq m_1 + m_2 + 1$ and $m_1 + 1 \leq j \leq m_2$. Thus, as argued above, $\ker(\tilde{a} - \lambda_2\, 1) = \mathrm{Span}\{\phi_{m_1+1}, \ldots, \phi_{m_1+m_2}\}$ is invariant under w and w^*. Hence, $wp_2 = p_2 w$, where p_2 is the projection of $\mathcal{H} \oplus \mathcal{H}$ onto $\ker(\tilde{a} - \lambda_2\, 1)$.

Now continue inductively and conclude that if $\lambda_1 > \lambda_2 > \cdots > 0$ are the positive eigenvalues of $\tilde{a}$, and if $p_1, p_2, \ldots$ are the corresponding projections onto the eigenspaces $\ker(\tilde{a} - \lambda_n\, 1)$, then $wp_n = p_n w$, for each n. Hence, if $\|\cdot\|$ denotes the norm in $B(\mathcal{H} \oplus \mathcal{H})$, we obtain from the compactness of $\tilde{a}$ that

$$\lim_n \|\tilde{a} - \sum_{j=1}^{n} \lambda_j p_j\| = 0,$$

and so

$$\begin{aligned}\|w\tilde{a} - \tilde{a}w\| &= \|w\tilde{a} - \sum_{j=1}^{n} \lambda_j w p_j + \sum_{j=1}^{n} \lambda_j p_j w - \tilde{a}w\| \\ &\leq \left\| w \left(\tilde{a} - \sum_{j=1}^{n} \lambda_j p_j \right) \right\| + \left\| \left(\tilde{a} - \sum_{j=1}^{n} \lambda_j p_j \right) w \right\|\end{aligned}$$

for every n, which implies that $\tilde{a}w - w\tilde{a} = 0$. Therefore, $\tilde{b}^q = w\tilde{a}^p w^* = \tilde{a}^p w w^* = \tilde{a}^p$. But $\tilde{a}^p = \tilde{b}^q$ if and only if $a^p = b^q$.

To finish the proof, we need to verify that $u^*ub = b$. We return to equation (2.1) to obtain

$$\mathrm{tr}\,(ub^q u^*) = \mathrm{tr}\,(b^q),$$

which we write as

$$\mathrm{tr}\,(b^{q/2} u^* u b^{q/2}) = \mathrm{tr}\,(b^q),$$

or equivalently as

$$\mathrm{tr}\,(b^{q/2}(1 - u^*u)b^{q/2}) = 0.$$

Because $1 - u^*u$ is a projection, we can rewrite the equation above as

$$\mathrm{tr}\,([(1 - u^*u)b^{q/2}]^*[(1 - u^*u)b^{q/2}]) = 0.$$

By faithfulness of the trace we conclude that $(1 - u^*u)b^{q/2} = b^{q/2}(1 - u^*u) = 0$. Thus, using functional calculus, we obtain $bu^*u = u^*ub = b$, implying that $R[u^*] = u^*u \geq R[b]$, which completes the proof. □

Note that in Theorem 2.1, the condition $a^p = b^q$ implies that $R[a] = R[b]$. But this is not the case when a, b are not positive, and indeed it is the projection $R[b]$ that is of interest in the extension below of Theorem 2.1 to arbitrary trace-class operators.

Theorem 2.2. *Let $a, b \in B(\mathcal{H})$ be trace-class operators and $u \in B(\mathcal{H})$ be a partial isometry. Suppose $p, q \in \mathbb{R}^+$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$tr\,(u|ab^*|u^*) = tr\left(\frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q\right) \tag{2.14}$$

if and only if

$$|b|^q = |a|^p, \quad \textit{and} \quad R[u^*] \geq R[b].$$

Proof. Note that if we consider the polar decomposition $v|b|$ of b, then from the proof of Proposition 4.1 in [EFZ] we have

$$|ab^*| = v\,\Big|\,|a||b|\,\Big|\,v^*.$$

Note also that v^*v is the projection onto the range of b^*.

To prove the sufficiency, assume that $|b|^q = |a|^p$ and $R[u^*] \geq R[b]$. Thus, $(b^*b)^q = (a^*a)^p$ and $u^*ub = b$, and therefore

$$\begin{aligned}\mathrm{tr}\,(u|ab^*|u^*) &= \mathrm{tr}\,\Big(uv\,\Big|\,|a||b|\,\Big|\,v^*u^*\Big) = \mathrm{tr}\,(uv|b|^q v^*u^*)\\ &= \mathrm{tr}\,(u^*ub|b|^{q-1}v^*) = \mathrm{tr}\,(b|b|^{q-1}v^*) = \mathrm{tr}\,(v|b|^q v^*)\\ &= \mathrm{tr}\,(v^*v|b|^q) = \mathrm{tr}\,(|b|^q)\\ &= \frac{1}{p}\,\mathrm{tr}\,(|a|^p) + \frac{1}{q}\,\mathrm{tr}\,(|b|^q).\end{aligned}$$

To prove the necessity, assume that equation (2.14) holds. Using twice the same idea as in (2.2), we obtain

$$\mathrm{tr}\,(u|ab^*|u^*) = \mathrm{tr}\,\left(uv\,\Big|\,|a||b|\,\Big|\,v^*u^*\right) \leq \mathrm{tr}\,\left(\Big|\,|a||b|\,\Big|\right).$$

The tracial Young inequality is

$$\mathrm{tr}\,\left(\Big|\,|a||b|\,\Big|\right) \leq \mathrm{tr}\,\left(\frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q\right),$$

(again from Proposition 3.3 of [EFZ]) and therefore

$$\mathrm{tr}\,\left(\Big|\,|a||b|\,\Big|\right) = \mathrm{tr}\,\left(\frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q\right).$$

Hence, by Theorem 2.1, we have that $|a|^p = |b|^q$. Therefore, the trace equation (2.14) simplifies to

$$\mathrm{tr}\,(uv|b|^q v^*u^*) = \mathrm{tr}\,(|b|^q). \tag{2.15}$$

Recall that $v|b| = b$ and $|b|v^* = b^*$, and so $v|b|^2v^* = bb^*$. Using this,

$$\mathrm{tr}\,(uv|b|^q v^* u^*) = \mathrm{tr}\,(u(bb^*)^{q/2}u^*). \tag{2.16}$$

Thus, noting that $\mathrm{tr}\,(|b|^q) = \mathrm{tr}\,((bb^*)^{q/2})$, equation (2.15) becomes

$$\mathrm{tr}\,((bb^*)^{q/4}(1 - u^*u)(bb^*)^{q/4}) = 0.$$

This equation and the faithfulness of the trace yield, as before,

$$(1 - u^*u)(bb^*)^{q/4} = 0.$$

Because $(bb^*)^{q/4}$ is positive, we apply functional calculus to get $(1-u^*u)bb^* = 0$. Then $(1 - u^*u)bb^*(1 - u^*u) = 0$, which implies that $(1 - u^*u)b = 0$; that is, $u^*ub = b$. □

3. Equality in the operator Young inequality

To characterise cases of equality in the operator Young inequality ([EFZ])

$$u|ab^*|u^* \le \frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q\,, \tag{3.1}$$

we shall employ the work in the previous section on the tracial inequality.

It is fairly easy to show that Young inequality in trace or Hilbert–Schmidt norm is strictly weaker than the operator inequality. Indeed, let a be any nonzero positive trace-class operator, let $b = a^{p/q}$, and let u be any unitary operator such that $ua \ne au$. Then $u^*ua = a$ and

$$\mathrm{tr}\,(u|ab^*|u^*) = \frac{1}{p}\,\mathrm{tr}\,(a^p) + \frac{1}{q}\,\mathrm{tr}\,(b^q);$$

but clearly

$$u|ab^*|u^* \ne \frac{1}{p}\,a^p + \frac{1}{q}\,b^q\,,$$

since the first term is ua^pu^* and the second term is a^p.

Furthermore, if a, b are trace-class operators and u is a unitary commuting with $|ab^*|$, then

$$\mathrm{tr}\,(u|ab^*|u^*) = \mathrm{tr}\,(|ab^*|) \le \frac{1}{p}\,\mathrm{tr}\,(|a|^p) + \frac{1}{q}\,\mathrm{tr}\,(|b|^q),$$

but $u|ab^*|u^* = |ab^*|$ can fail to be dominated as an operator by $\frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q$, even if the Hilbert space has finite dimension.

As before, we examine the case of positive operators first.

Theorem 3.1. *Let $a, b \in B(\mathcal{H})$ be positive trace-class operators, and suppose that $u \in B(\mathcal{H})$ is a partial isometry. Let $p, q \in \mathbb{R}^+$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements are equivalent:*

1. $u|ab|u^* = \dfrac{1}{p}\, a^p + \dfrac{1}{q}\, b^q$;
2. $b^q = a^p$, $R[u^*] \geq R[b]$, *and* $ub = bu$.

Proof. Write $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_0$, where $\mathcal{H}_0 = \ker b = \ker b^q$. Thus,

$$b = \begin{bmatrix} b_+ & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \tag{3.2}$$

where $\ker b_+ = \ker b_+^q = \{0\}$.

To prove that (1) implies (2), assume that $u|ab|u^* = \dfrac{1}{p}\, a^p + \dfrac{1}{q}\, b^q$. Then, of course,

$$\operatorname{tr}(u|ab|u^*) = \frac{1}{p} \operatorname{tr}(a^p) + \frac{1}{q} \operatorname{tr}(b^q),$$

which implies by Theorem 2.1 that $b^q = a^p$ and $R[u^*] \geq R[b]$. Thus, it remains to show that u and b commute.

The operator equation $u|ab|u^* = \dfrac{1}{p}\, a^p + \dfrac{1}{q}\, b^q$ simplifies to

$$u b^q u^* = b^q\,, \tag{3.3}$$

which indicates that $\ker b^q = \ker b$ is invariant under u^*. Hence, $u_{12} = 0$. Therefore, the matricial form of (3.3) is

$$\begin{bmatrix} b_+^q & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u_{11} b_+^q u_{11}^* & u_{11} b_+^q u_{21}^* \\ u_{12} b_+^q u_{11}^* & u_{21} b_+^q u_{21}^* \end{bmatrix}. \tag{3.4}$$

The $(2,2)$-entries in (3.4) show that $\langle b_+^q u_{21}^* \xi_0, u_{21}^* \xi_0 \rangle = 0$, for all $\xi_0 \in \mathcal{H}_0$. Hence, u_{21}^* maps $\mathcal{H}_0$ into $\ker b_+^q = \{0\}$, implying that $u_{21} = 0$. So, $u = u_+ \oplus u_0$, where $u_+ = u_{11}$ and $u_0 = u_{22}$.

Now equating the $(1,1)$-entries in (3.4), we have $u_+ b_+^q u_+^* = b_+^q$. Thus, $\overline{\operatorname{ran} u_+} \supseteq \overline{\operatorname{ran} b_+} = \mathcal{H}_+$, meaning that u_+ has dense range. Furthermore, the condition $R[u^*] \geq R[b]$ implies that $R[u_+^*] \geq R[b_+] = 1_{\mathcal{H}_+}$. Hence, $u_+^* u_+ = 1_{\mathcal{H}_+}$ and so u_+ is a surjective isometry (i.e., a unitary). Therefore, u_+ commutes with b_+^q. Passing to the q-th root, u_+ commutes with b_+ and therefore $u = u_+ \oplus u_0$ commutes with $b = b_+ \oplus 0$.

To prove that (2) implies (1), assume that $b^q = a^p$, $R[u^*] \geq R[b]$, and $ub = bu$. The matrix forms (3.2) of b and u satisfy $ub = bu$ only if $b_+ u_{12} = 0$ and $u_{21} b_+ = 0$. But $\ker b_+ = \{0\}$ shows that $b_+ u_{12} = 0$ is possible only if $u_{12} = 0$. Likewise, passing to the adjoint of $u_{21} b_+ = 0$, the equation $b_+ u_{21}^* = 0$ holds only if $u_{21}^* = 0$. Therefore, $u = u_+ \oplus u_0$, where $u_+ = u_{11}$ and $u_0 = u_{22}$.

As noted above, the assumption that $R[u^*] \geq R[b]$ implies $R[u_+^*] \geq R[b_+]$ and, hence, u_+ is an isometry. But moreover, $ub = bu$ implies that $u_+b_+ = b_+u_+$, and so the range of the isometry u_+ contains the closed range of b_+, namely $\mathcal{H}_+$. Thus, u_+ is a unitary commuting with b_+ and so $u_+b_+^q u_+^* = b_+^q$, which clearly implies that $ub^q u^* = b^q$. Now making use of $b^q = a^p$, we conclude that $u|ab|u^* = \frac{1}{p}\, a^p + \frac{1}{q}\, b^q$. □

Now we extend Theorem 3.1 to arbitrary trace-class operators. To do so we shall make repeated use of the equation

$$|ab^*| = v \left| \, |a||b| \, \right| v^* \,,$$

where $b = v|b|$ is the polar decomposition of b.

Theorem 3.2. *Let $a, b \in B(\mathcal{H})$ be trace-class operators, and suppose that $u \in B(\mathcal{H})$ is a partial isometry. Let $p, q \in \mathbb{R}^+$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements are equivalent:*

1. *$u|ab^*|u^* = \frac{1}{p}\, |a|^p + \frac{1}{q}\, |b|^q$;*
2. *$|b|^q = |a|^p$, $R[u^*] \geq R[b]$, and $ub = v^*(bu)v$,*

where $b = v|b|$ is the polar decomposition of b.

Proof. Assume first that $|b|^q = |a|^p$, $R[u^*] \geq R[b]$, and $ub = v^*(bu)v$. Note that uv is a partial isometry. Indeed, since $vv^* = R[b]$, we have that $u^*uv = v$, or equivalently $v^*u^*u = v^*$. So $v^*u^*uv = v^*v$ is a projection, and this implies that uv is a partial isometry.

The condition $u^*ub = b$ can be written as $u^*uv|b| = v|b|$. If we apply v^* to this equality and take into account that $v^*v|b| = |b|$, then $(uv)^*(uv)|b| = |b|$. The condition $ub = v^*buv$ can be written as $uv|b| = |b|uv$. So we have shown that the conditions (2) are equivalent to

$$|b|^q = |a|^p, \quad R[(uv)^*] \geq R[|b|], \quad (uv)b = b(uv),$$

with uv a partial isometry. Then, from Theorem 3.1, we get that

$$uv \left| \, |a||b| \, \right| v^*u^* = \frac{1}{p}\, |a|^p + \frac{1}{q}\, |b|^q,$$

that is

$$u|ab^*|u^* = \frac{1}{p}\, |a|^p + \frac{1}{q}\, |b|^q.$$

Conversely, if we assume condition (1), we can write it as

$$uv \left| \, |a|\, |b| \, \right| v^*u^* = \frac{1}{p}\, |a|^p + \frac{1}{q}\, |b|^q, \tag{3.5}$$

where $b = v|b|$ is the polar decomposition of b. From Theorem 2.2 applied to condition (1), we have that $|b|^q = |a|^p$, and $R[u^*] \geq R[b]$. Then again we know that $u^*uv = v$, and so uv is a partial isometry, as we proved in the first paragraph. Now from equation (3.5) and Theorem 3.1, we have that $R[(uv)^*] \geq R[b]$ and $uv|b| = |b|uv$. As in the first part of the proof, these relations are easily seen to be equal to the ones in condition (2). □

4. Remarks and applications

This work began with the goal of proving that equality in the operator Young inequality

$$u|ab^*|u^* \leq \frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q, \tag{4.1}$$

implies that $|a|^p = |b|^q$. While this problem remains open for arbitrary compact operators, it has been settled by Hirzallah and Kittaneh (in Remark 4, following Corollary 3 of [HK]) for Hilbert–Schmidt operators by way of the remarkable inequality

$$\|ab^*\|_2^2 + \frac{1}{r^2}\|\,|a|^p - |b|^q\,\|_2^2 \leq \left\| \frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q \right\|_2^2,$$

where $r = \max\{p, q\}$. Apart from the fact that the inequality itself is notable, there is something else that the inequality leads to: the surprising fact that to consider the case of equality in the operator inequality (4.1), one need only assume that there is equality in the Hilbert–Schmidt norm. Our results in the present paper are in the same vein: it is sufficient to assume only that $\operatorname{tr}(u|ab^*|u^*) = \frac{1}{p}\operatorname{tr}(|a|^p) + \frac{1}{q}\operatorname{tr}(|b|^q)$.

Combining [HK] with Theorem 2.2, one has the following result, which shows the surprising "rigidity" of the Young inequality.

Theorem 4.1. *Suppose $p, q \in \mathbb{R}^+$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. If $a, b \in B(\mathcal{H})$ are trace-class operators and if $u \in B(\mathcal{H})$ is a partial isometry, then the following conditions are equivalent:*

1. $\|u|ab^*|u^*\|_1 = \left\| \frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q \right\|_1$;
2. $\|u|ab^*|u^*\|_2 = \left\| \frac{1}{p}\,|a|^p + \frac{1}{q}\,|b|^q \right\|_2$;
3. $|a|^p = |b|^q$ *and* $R[u^*] \geq R[b]$.

Proof. The only thing to be verified is that one can eliminate from consideration the partial isometry u in (2). This can be done by using the arguments employed in (2.2) and (2.3), but applying them to the Hilbert–Schmidt norm instead of the trace. And then apply twice a reasoning like the one at the end of the proof of Theorem 2.2 to get that $R[u^*] \geq R[b]$. □

The rigidity condition mentioned in the Theorem 4.1 is in fact stronger than the way it was stated. In particular, for matrices a rather complete statement can be established:

Theorem 4.2. *Suppose $p, q \in \mathbb{R}^+$ are such that $\frac{1}{p} + \frac{1}{q} = 1$. If $a, b \in M_n(\mathbb{C})$ are matrices and if $u \in M_n(\mathbb{C})$ is a partial isometry, then the following conditions are equivalent:*

1. $|||u|ab^*|u^*||| = \left|\left|\left| \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \right|\right|\right|$ *for every unitarily invariant norm* $||| \cdot |||$;
2. $|a|^p = |b|^q$ *and* $R[u^*] \geq R[b]$.

Proof. The assertion (1) implies (2) is clear from Theorem 4.1, for the trace norm is a unitarily invariant norm.

Conversely, if $|a|^p = |b|^q$, then $s_j(a)^p = s_j(b)^q$ for every $j = 1, \ldots, n$, where s_j denotes the j-th singluar value. We write $b = v|b|$ the polar decomposition of b and note, as in the proof of Theorem 3.2, that $u^*uv = v$ and $v^*v|b| = |b|$. Then, writing $\|\cdot\|_{(k)}$ for the k-th Ky Fan norm, we have (using that $s_j(x^*x) = s_j(xx^*)$ for every j)

$$\begin{aligned} \|u|ab^*|u^*\|_{(k)} &= \sum_{j=1}^{k} s_j(u|ab^*|u^*) = \sum_{j=1}^{k} s_j(uv|b|^q v^* u^*) \\ &= \sum_{j=1}^{k} s_j(|b|^{q/2} v^* u^* uv |b|^{q/2}) = \sum_{j=1}^{k} s_j(|b|^{q/2} v^* v |b|^{q/2}) \\ &= \sum_{j=1}^{k} s_j(|b|^q) = \sum_{j=1}^{k} s_j\left(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q\right) \\ &= \left\| \frac{1}{p}|a|^p + \frac{1}{q}|b|^q \right\|_{(k)}. \end{aligned} \tag{4.2}$$

Thus, we have equality for all the Ky Fan norms, and so we have equality for every unitarily invariant norm (see Theorem IV.2.2 of [Ba]). □

Let us also mention the fact that one cannot expect in general that equality in just **one** unitarily invariant norm forces condition (2) in Theorem 4.2. For instance one can consider the Ky Fan norm $\|\cdot\|_{(1)}$ (that is, the operator norm). Let

$$a = \begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then clearly

$$\||ab|\| = 1, \quad \left\| \frac{1}{p} a^p + \frac{1}{q} b^q \right\| = 1,$$

but $a^p \neq b^q$. With the same idea it is easy to build examples for all the k-th Ky Fan norms, $k < n$. (However, the case $k = n$ is the trace norm.)

While working on this paper, the authors frequently encountered problems concerning where the involution $*$ ought to be placed in conditions like $R[u^*] \geq R[b]$. This difficulty is an illustration of an insightful and eloquent comment of R.C. Thompson (at the bottom of page 87 of [T]) on the challenges one faces in formulating operator inequalities. With Young's inequality, at least for positive operators, in the end it so happens that the concern about where to place the $*$ really does not matter. Indeed, this is because the condition of equality is so strong: if we have trace-class operators $a, b \geq 0$ and a partial isometry u such that

$$u|ab|u^* = \frac{1}{p} a^p + \frac{1}{q} b^q ,$$

then by Theorem 3.1 we know that $a^p = b^q$, $R[u^*] \geq R[b]$ and $ub = bu$. We have that

$$ub^q u^* = b^q .$$

But, since $u^*ub = b$, if we conjugate the above equation with u^* and u, we get

$$b^q = u^* b^q u,$$

which implies that

$$u^*|ab|u = \frac{1}{p} a^p + \frac{1}{q} b^q .$$

That is, in the operator equality the roles of u and u^* can be interchanged.

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