LOCAL MULTIPLIER ALGEBRAS, INJECTIVE ENVELOPES, AND TYPE I \( W^* \)-ALGEBRAS

MARTÍN ARGERAMI AND DOUGLAS R. FARENICK

Abstract. Characterizations of those separable C*-algebras that have \( W^* \)-algebra injective envelopes or \( W^* \)-algebra local multiplier algebras are presented. The C*-envelope and the injective envelope of a class of operator systems that generate certain type I von Neumann algebras are also determined.

The local multiplier algebra \( M_{\text{loc}}(A) \) of a C*-algebra \( A \) is the C*-algebraic direct limit of multiplier algebras \( M(K) \) along the downward-directed system \( E(A) \) of all (closed) essential ideals \( K \) of \( A \). Such algebras first arose in the study of derivations and were formally introduced by Pedersen in [17], where he proves that every derivation on a separable C*-algebra \( A \) extends to an inner derivation of \( M_{\text{loc}}(A) \). The question of whether every derivation of \( M_{\text{loc}}(A) \) is inner remains open for arbitrary separable C*-algebras.

A systematic study of local multiplier algebras is presented in the recent monograph by Ara and Mathieu [2]. One of the most important general facts concerning local multiplier algebras is that the centre \( Z(M_{\text{loc}}(A)) \) of \( M_{\text{loc}}(A) \) is an AW*-algebra [1]. Although \( M_{\text{loc}}(A) \) itself need not be an AW*-algebra, Frank and Paulsen [8] have showed recently that \( M_{\text{loc}}(A) \) can nevertheless be realized as a C*-subalgebra of a certain minimal injective AW*-algebra: namely, the injective envelope \( I(A) \) of \( A \) [9]. Further, even though \( M_{\text{loc}}(A) \) is not in general an AW*-algebra, there are examples in which \( M_{\text{loc}}(A) \) is actually a \( W^* \)-algebra. We show herein that for separable C*-algebras, \( M_{\text{loc}}(A) \) is a \( W^* \)-algebra if and only if \( A \) has a minimal essential ideal that is isomorphic to a C*-algebraic direct sum of elementary C*-algebras. This result also leads to a new proof of a theorem arising from work of Wright [20] and Hamana [12] that characterizes those separable \( A \) for which \( I(A) \) is a \( W^* \)-algebra.

As usual, we will denote by \( B(H) \) and \( K(H) \) the set of bounded and compact operators on a Hilbert space \( H \).

The notion of injective envelope [9, 10, 16] first arose in two seminal papers of Arveson [4, 5]. One of the principal results of [5], the so-called boundary theorem, states that if \( E \) is an operator system acting on a Hilbert space \( H \) such that \( K(H) \subset C^*(E) \), then the identity map on \( E \) has a unique completely positive extension to the algebra \( C^*(E) \subset B(H) \) if and only if the quotient homomorphism onto the Calkin algebra is not completely isometric on \( E \). This theorem is revisited in the present paper for a class of operator systems that generate discrete type I von Neumann algebras.

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Let $\mathcal{E}(A)$ denote the set of (closed) essential ideals of a $C^*$-algebra $A$. For every $K \in \mathcal{E}(A)$, let $M(K)$ denote the multiplier algebra of $K$. If $K_1, K_2 \in \mathcal{E}(A)$ are such that $K_1 \subseteq K_2$, then $M(K_1) \supseteq M(K_2)$; thus, the family $\mathcal{E}(A)$ of essential ideals of $A$ determines a downward-directed system of $C^*$-algebras. The local multiplier algebra $M_{\text{loc}}(A)$ of $A$ is $C^*$-algebraic direct limit that arises from $\mathcal{E}(A)$:

$$M_{\text{loc}}(A) = \lim_{\rightarrow} \{ M(K) : K \in \mathcal{E}(A) \}.$$  

Every $C^*$-algebra $A$ is a $C^*$-subalgebra of its injective envelope $I(A)$ [9]. Moreover, by [8, Corollary 4.3],

$$M_{\text{loc}}(A) = \left( \bigcup_{K \in \mathcal{E}(A)} \{ x \in I(A) : xK + Kx \subseteq K \} \right)^{-},$$

where the closure is with respect to the norm topology of $I(A)$. Thus,

$$A \subseteq M_{\text{loc}}(A) \subseteq I(A)$$

is an inclusion of $C^*$-subalgebras. In [7], Frank showed an additional sequence of inclusions as $C^*$-subalgebras:

$$A \subseteq M_{\text{loc}}(A) \subseteq M_{\text{loc}}(M_{\text{loc}}(A)) \subseteq \overline{A} \subseteq I(A).$$

In the inclusions above, $\overline{A}$ is the regular monotone completion [11] of $A$. For separable $C^*$-algebras, $\overline{A}$ coincides with $\overline{A}^\sigma$, the regular monotone $\sigma$-completion [19] of $A$.

It is not known whether $\overline{A} \neq I(A)$ for separable $C^*$-algebras $A$, but all other inclusions above can be proper. Most striking is the recent example of Ara and Mathieu [3] in which they show that $M_{\text{loc}}(A) \neq M_{\text{loc}}(M_{\text{loc}}(A))$ for a certain prime AF $C^*$-algebra $A$.

Further relations are: $I(M_{\text{loc}}(A)) = I(A)$ [8, Theorem 4.6] and $\mathcal{Z}(M_{\text{loc}}(A)) = M_{\text{loc}}(\mathcal{Z}(A)) = \mathcal{Z}(I(A))$ [7, Theorem 2] (since $\mathcal{Z}(M_{\text{loc}}(A))$ is an AW$^*$-algebra [2, Proposition 3.1.5] and, as it is abelian, is therefore injective).

1. $M_{\text{loc}}(A)$ as a $W^*$-algebra

It need not be true that $M_{\text{loc}}(A)$ is an AW$^*$-algebra. For example, $M_{\text{loc}}(A) = A$ in the case where $A$ is unital, simple, and separable—but AW$^*$-algebras (of infinite dimension) are nonseparable. Although it is even less likely that $M_{\text{loc}}(A)$ is a $W^*$-algebra, this is precisely the case for a number of important examples (such as if $A$ is a von Neumann algebra or if $A$ can be represented as acting on a Hilbert space $H$ in such a way as to contain every compact operator).

Theorem 1.2 below characterizes those separable $C^*$-algebras that admit $W^*$-algebra local multipliers. To prepare the way, the following lemma will be of use.

**Lemma 1.1.** If $A$ is a separable $C^*$-algebra and if $M_{\text{loc}}(A)$ is a $W^*$-algebra, then $M_{\text{loc}}(A)$ is of type I.
Proof. Without loss of generality, we assume that \( M_{\text{loc}}(A) \) is faithfully represented as a von Neumann algebra acting on a Hilbert space \( H \). Thus,

\[
A \subseteq A'' \subseteq M_{\text{loc}}(A) \subseteq \overline{A} \subseteq I(A),
\]
as an inclusion of operator systems. Because \( A \) is order dense in \( \overline{A} \) [11], \( A \) is also order dense in \( A'' \); that is,

\[
h = \sup \{ k \in A^+ \mid k \leq h \}, \quad \forall h \in A''.
\]
Consequently, for any normal state \( \omega \) on \( A'' \), \( \omega(h) \geq \sup \{ \omega(k) \mid k \in A^+, \ k \leq h \} \).
Hence, any normal state \( \omega \) on \( A'' \) that is faithful on \( A \) is also faithful on \( A'' \). This implies, by a theorem of Takesaki [18], that \( A'' \) is generated by its minimal projections and each minimal projection of \( A'' \) is contained in \( A \). Hence, \( A'' \) is a discrete type I von Neumann algebra. Since type I AW\(^*\)-algebras are injective, we conclude that \( A'' = M_{\text{loc}}(A) = I(A) \).
\( \square \)

We shall employ the following notation from [2]. If \( \{ A_\alpha \}_{\alpha \in \Lambda} \) is a family of \( C^* \)-algebras, then

\[
\prod_{\alpha \in \Lambda} A_\alpha = \{ (a_\alpha)_{\alpha} : a_\alpha \in A_\alpha \text{ and } \sup_{\alpha} \| a_\alpha \| < \infty \} ;
\]
\[
\bigoplus_{\alpha \in \Lambda} A_\alpha = \{ (a_\alpha)_{\alpha} : a_\alpha \in A_\alpha \text{ and } \forall \varepsilon > 0 \text{ only finitely many } a_\alpha \text{ satisfy } \| a_\alpha \| > \varepsilon \}.
\]
Note that the direct product \( \prod_{\alpha} A_\alpha \) and the direct sum \( \bigoplus_{\alpha} A_\alpha \) are \( C^* \)-algebras and \( \bigoplus_{\alpha} A_\alpha \) is an ideal of \( \prod_{\alpha} A_\alpha \).

The next theorem is one of the main results of this paper. Recall that an elementary \( C^* \)-algebra is one that is isomorphic to \( K(H) \) for some Hilbert space \( H \).

**Theorem 1.2.** The following statements are equivalent for a separable \( C^* \)-algebra \( A \).

1. \( I(A) \) is a \( \mathcal{W}^* \)-algebra.
2. \( M_{\text{loc}}(A) \) is a \( \mathcal{W}^* \)-algebra.
3. \( M_{\text{loc}}(A) \) is a discrete type I \( \mathcal{W}^* \)-algebra.
4. \( A \) contains a minimal essential ideal that is isomorphic to a direct sum of elementary \( C^* \)-algebras.

**Proof.** (1) \( \Rightarrow \) (4). As \( A \) is a \( C^* \)-algebra whose injective envelope \( I(A) \) is a \( \mathcal{W}^* \)-algebra, \( \overline{A} \) is also a \( \mathcal{W}^* \)-algebra (because a monotone closed \( C^* \)-subalgebra of a von Neumann algebra is a von Neumann algebra [13]). Since \( A \) is separable, \( \overline{A} \) has a countable order-dense subset (Wright notes in [21, page 84] that the equivalence of the separability and having a countable order-dense subset follows from Theorem 4.3 of [19]). Hence, by [21, Proposition A], the set of pure states of \( \overline{A} \) (in the weak* topology) is hyperseparable. Since hyperseparability implies separability, another theorem of Wright [20, Corollary 7] shows that \( \overline{A} \) is isomorphic to \( \prod_{n} B(H_n) \) (a countable product), with each \( H_n \) separable. Further, since \( \prod_{n} B(H_n) \) is injective, it follows that \( I(A) = \overline{A} = \prod_{n} B(H_n) \). Finally, Lemma 3.1(iii) of [12] yields that \( \bigoplus_{n} K(H_n) \subseteq A \subseteq \prod_{n} B(H_n) \). The minimality of \( \bigoplus_{n} K(H_n) \) is given by [12, Proposition 3.3].
(4) \( \Rightarrow \) (3). Suppose that \( A \) has a minimal essential ideal \( K \) such that \( K \cong \bigoplus_n K(H_n) \). Therefore, by [2, Lemma 1.2.1],

\[
M(K) = M \left( \bigoplus_n K(H_n) \right) = \prod_n M(K(H_n)) = \prod_n B(H_n),
\]

which shows that \( M(K) \) is a (discrete type I) \( W^* \)-algebra. Furthermore, because \( K \) is a minimal essential ideal of \( A \), \( M(K) = M_{\text{loc}}(A) \) by [2, Remark 2.3.7]. Hence, \( M_{\text{loc}}(A) \) is a discrete type I \( W^* \)-algebra.

The implication (3) \( \Rightarrow \) (2) is trivial, and the implication (2) \( \Rightarrow \) (3) is Lemma 1.1. (3) \( \Rightarrow \) (1). Type I \( AW^* \)-algebras are injective, and so the inclusion \( A \subseteq M_{\text{loc}}(A) \subseteq I(A) \) (with \( M_{\text{loc}}(A) \) injective) implies that \( M_{\text{loc}}(A) = I(A) \), whence \( I(A) \) is a \( W^* \)-algebra.

\[ \square \]

**Corollary 1.3.** If any one of the equivalent conditions in Theorem 1.2 holds for a separable \( C^* \)-algebra \( A \), then

\[
M_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A)) = \overline{A} = I(A).
\]

**Proof.** Assume that any one of the statements (1)–(4) in Theorem 1.2 holds. Then \( M_{\text{loc}}(A) \) is an injective \( W^* \)-algebra. However, \( A \subseteq M_{\text{loc}}(A) \subseteq I(A) \) as \( C^* \)-subalgebras, and so by definition of the injective envelope, it must be that \( M_{\text{loc}}(A) = I(A) \), which proves that \( M_{\text{loc}}(A) = M_{\text{loc}}(M_{\text{loc}}(A)) = \overline{A} = I(A) \).

\[ \square \]

**Corollary 1.4.** If \( A \) is a separable, prime \( C^* \)-algebra, then exactly one of the following two statements holds:

1. \( I(A) \cong B(H) \), for some separable Hilbert space \( H \);
2. \( I(A) \) is a wild type III \( AW^* \)-factor.

In particular, if \( A \) has no nonzero postliminal ideals, then \( I(A) \) is a wild type III \( AW^* \)-factor.

**Proof.** Because \( A \) is prime, \( I(A) \) is an \( AW^* \)-factor [11, Theorem 7.1]. This factor cannot be of type II for the following reasons.

If \( I(A) \) is a finite type II \( AW^* \)-factor, then the identity \( 1 \in I(A) \) is a finite projection, and so \( 1 \) is a finite projection in \( \overline{A} \) as well. Therefore, \( \overline{A} \) is of type I [15, Theorem 2]. But type I algebras are injective; hence \( \overline{A} = I(A) \), contradicting that \( I(A) \) is of type II. Thus, assume that \( I(A) \) is a type \( II_{\infty} \) \( AW^* \)-factor. Since \( I(A) \) admits a faithful state (because \( A \) is separable), \( I(A) \) is a \( W^* \)-algebra by [6]. So Theorem 1.2 implies that \( I(A) \) is of type I, which is a contradiction. Hence, \( I(A) \) is a factor of either type I or type III.

In the case where \( I(A) \) is of type I we have \( I(A) \cong B(H) \) for some Hilbert space \( H \), because all type I \( AW^* \)-factors have this form [14, Theorem 2]. Indeed, in this case, \( \overline{A} = I(A) \cong B(H) \); since \( \overline{A} \) is countably decomposable, \( H \) can be chosen to be separable.

If \( I(A) \) is not of type I, then the type III \( AW^* \)-factor \( I(A) \) cannot be a \( W^* \)-algebra, by Theorem 1.2. Every \( AW^* \)-factor that is not \( W^* \)-algebra is wild [21]; hence, \( I(A) \) is wild.
Finally, if \( A \) is prime and has a nonzero postliminal ideal, then \( I(A) \) is of type I \([12]\). Thus, a prime separable C*-algebra with no nonzero postliminal ideals must have a wild type III injective envelope. \( \Box \)

2. A Version of the Boundary Theorem

If \( E \) is an operator system, then the C*-envelope \([10, 16]\) of \( E \) is the C*-subalgebra \( C^*_\text{env}(E) \) of \( I(E) \) generated by \( E \). The C*-algebra \( C^*_\text{env}(E) \) is independent of the choice embedding of \( E \) into an injective envelope \((I, \kappa)\) of \( E \); thus, the notation \( C^*_\text{env}(E) \) is unambiguous.

The aim of the present section is to prove the following result.

**Theorem 2.1.** Let \( E \subseteq B(H) \) be an operator system for which the von Neumann algebra \( E'' \) is generated by its minimal projections, each of which is contained in the C*-subalgebra \( C^*(E) \) of \( B(H) \) generated by \( E \). Then \( I(E) \) is a type I W*-algebra and

\[
I(E) \cong E'' \quad \text{and} \quad C^*_\text{env}(E) \cong C^*(E).
\]

Before turning to the proof of Theorem 2.1, recall that the original motivation for the concept of injectivity is Arveson’s Hahn–Banach Extension Theorem \([4]\) for completely positive linear maps, and that the idea of an injective envelope stems from Arveson’s theory of boundary representations \([5]\). In Arveson’s work on boundary representations, the operator systems were often realized as irreducible operator systems in \( B(H) \) and their generated C*-algebras \( C^*(E) \) were sometimes assumed to have nontrivial intersection with—and hence contain—the ideal \( K(H) \) of compact operators. In this spirit, Theorem 2.1 is a generalization of the boundary theorem from \( B(H) \) to discrete type I von Neumann algebras.

Two preliminary results are needed for the proof of Theorem 2.1. The first result is a proposition of Hamana that is a useful criterion for determining when an injective operator system \( I \) containing \( E \) is an injective envelope.

**Proposition 2.2.** (\([9, \text{Lemma } 3.7]\)) Consider an inclusion \( E \subseteq I \) of operator systems, where \( I \) is injective. Then the following statements are equivalent.

1. \( I \) is an injective envelope of \( E \).
2. The only completely positive linear map \( \omega : I \to I \) for which \( \omega|_E = \text{id}_E \) is the identity map \( \omega = \text{id}_I \).

The second preliminary result is a kind of partial converse to the main result of \([18]\).

**Lemma 2.3.** Suppose that \( A \) is a C*-subalgebra of a von Neumann algebra \( M \) and that \( M = A'' \). If \( M \) is generated by its minimal projections, each of which is contained in \( A \), then \( A \) is order dense in \( M \).

**Proof.** Choose a nonzero \( h \in M^+ \) and consider the set

\[
\mathcal{F} = \{ (k_i) \subset A^+ : \sum_{\text{finite}} k_i \leq h \}.
\]
There is a strictly positive \( \lambda \) in the spectrum \( \sigma(h) \) of \( h \). Let \( e \in M \) be the spectral projection \( e = e^h((\lambda, \infty)) \), where \( e^h \) denotes the spectral resolution of \( h \). Thus, \( 0 \neq \lambda e \leq he \). Moreover, \( e \) majorizes a minimal projection \( p \) of \( M \); by hypothesis, \( p \in A \). Thus, \( 0 \neq \lambda p = e(\lambda p)e \leq e(\lambda)e = \lambda e \leq he \leq h \), and so \( \lambda p \in \mathcal{F} \), which proves that \( \mathcal{F} \neq \emptyset \). It is clear that \( \mathcal{F} \) is inductive under inclusions of those families and so, by Zorn’s Lemma, \( \mathcal{F} \) has a maximal family \( W \). Since every finite sum of this family is less than \( h \),

\[
y = \sup \left\{ \sum_{k \in K} k : K \text{ is finite and } K \subset W \right\} \leq h.
\]

If \( y \neq h \), then \( h - y \in M^+ \), and so by the first paragraph there exists nonzero \( k \in A^+ \) such that \( k \leq h - y \). If it were true that \( k \in W \), then for each net \( (h_i) \) of those finite sums of elements in \( W \) such that \( h_i \not\succ y+k \), the net \( (h_i+k) \not\succ y+k \), which contradicts the fact that \( y \) is the supremum. Hence, \( k \not\in W \). But if \( k \not\in W \), then the family \( W \) is not maximal, which is again a contradiction. Therefore, it must be that \( y = h \), which proves that \( A \) is order dense in \( M \).

**Theorem 2.4.** If \( A \subseteq B(H) \) is a C*-algebra and if \( M = A'' \) is generated by its minimal projections, each of which is contained in \( A \), then \( \varphi = \text{id}_M \) for every completely positive linear map \( \varphi : M \to M \) for which \( \varphi|_A = \text{id}_A \).

**Proof.** Observe that because \( \varphi : M \to M \) is a unital completely positive map that preserves \( A \), \( \varphi \) has the following property:

\[
\varphi(xk) = \varphi(x)k, \text{ for every } k \in A.
\]

This fact follows from the Cauchy-Schwarz inequality and from the fact that \( A \) is in the multiplicative domain of \( \varphi \) [16, Theorem 3.18]. Using this fact we shall deduce below that

\[
(2.1) \quad x \geq 0 \text{ if and only if } \varphi(x) \geq 0.
\]

Indeed, one implication is obvious from the positivity of \( \varphi \). To prove the other implication, assume that \( \varphi(x) \geq 0 \). Thus, \( \text{Im} (\varphi(x)) = \varphi(\text{Im} (x)) = 0 \). Let \( z = \text{Im} (x) \) and write \( z = z^+ - z^- \), where \( z^+, z^- \in M^+ \) are such that \( z^+z^- = z^-z^+ = 0 \).

Our first goal is to prove that \( z^+ = 0 \). Suppose, on the contrary, that \( z^+ \neq 0 \). Thus, there is a strictly positive \( \lambda \) in the spectrum of \( z^+ \); hence, there is a spectral projection \( p \in M \) such that \( 0 \neq \lambda p \leq pz^+ = z^+p \). Note that \( z^-p = 0 \), as the projection \( p \) is in the von Neumann algebra generated by \( z^+ \) and \( z^+z^- = z^-z^+ = 0 \). Let \( q \in A \) be an arbitrary minimal projection of \( M \) and consider the projection \( p \wedge q \in M \). Because \( p \wedge q \leq q \) and \( q \) is minimal, either \( p \wedge q = 0 \) or \( p \wedge q = q \). We will show that the latter case cannot occur (under the conventional assumption that minimal projections are defined to be nonzero). Assume that it is true that \( p \wedge q = q \). Then \( 0 \neq q = p \wedge q \leq p \). Pre- and post-multiply the inequality \( \lambda q \leq \lambda p \leq z^+p = zp \) by \( q \) to obtain \( \lambda q \leq q(zp)q \leq qzq \). Note that \( \varphi(qzq) = \varphi(zq) = q\varphi(z)q = 0 \) and \( 0 \leq \lambda q = \varphi(\lambda q) \leq q\varphi(z)q = 0 \). This implies that \( q = 0 \), which contradicts the
fact that $q$ is minimal and, thus, nonzero. Therefore, it must be that $p \wedge q = 0$, for every minimal projection $q$ of $M$. Because every nonzero projection in $M$ majorizes a minimal projection, we conclude that $p = 0$, in contradiction to the fact that $p$ is a nonzero spectral projection of $z^+$. Hence, it must be that $z^+ = 0$.

A similar argument shows that $z^- = 0$. We can find a nonzero $\lambda \in \mathbb{R}^+$ and a minimal projection $q \in A$ such that $qzq \leq -\lambda q$; thus $-\lambda q = \varphi(-\lambda q) \geq \varphi(qzq) = q\varphi(z)q = 0$, and again $q = 0$.

We conclude that $z = 0$, which implies that $x$ is selfadjoint. It remains to show that $x$ is positive. Assume that $x$ is not positive. Thus, there exists a nonzero spectral projection in the negative part of $\sigma(x)$; by taking once again a suitable minimal subprojection $q$, we can find $\lambda > 0$ such that $qzxq \leq -\lambda q$. But then $\varphi(qzxq) \leq -\lambda q$; and on the other hand, $\varphi(qzxq) = q\varphi(x)q \geq 0$. The contradiction implies that no such $q$ can exist, and so $x \geq 0$.

From (2.1) and the fact the $\varphi$ preserves $A$, we have that for $k \in A$, $k \leq x$ if and only if $k \leq \varphi(x)$. Lemma 2.3 asserts that $A$ is order dense in $M$. Hence, $\varphi(x) = x$ for every $x \in M^+$, which implies that $\varphi$ is the identity map on $M$. \hfill \qed

Proof of Theorem 2.1. By hypothesis, $C^*(E)$ contains all the minimal projections that generate $E''$. Theorem 2.4 together with Proposition 2.2 show that $E''$ is an injective envelope for $E$. Further, there is a completely positive projection $\phi$ on $B(H)$ with range $E''$. Hence, if $x, y \in E''$, then $x \circ y$—the product of $x$ and $y$ in the $C^*$-algebra $I(E)$—is given by $x \circ y = \phi(xy) = xy$, since $E''$ is an algebra. Thus, $E'' = I(E)$ and $C^*(E)$ is precisely $C^*_{env}(E)$. \hfill \qed

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References


Department of Mathematics and Statistics
University of Regina
Regina, Saskatchewan
Canada S4S 0A2

E-mail address:argerami@math.uregina.ca
E-mail address:farenick@math.uregina.ca