LOCAL MULTIPLIER ALGEBRAS, INJECTIVE ENVELOPES, AND TYPE I W*-ALGEBRAS

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ABSTRACT. Characterizations of those separable C*-algebras that have W*-algebra injective envelopes or W*-algebra local multiplier algebras are presented. The C*-envelope and the injective envelope of a class of operator systems that generate certain type I von Neumann algebras are also determined.

The local multiplier algebra $M_{\rm loc}(A)$ of a C*-algebra A is the C*-algebraic direct limit of multiplier algebras $M(K)$ along the downward-directed system $\mathcal{E}(A)$ of all (closed) essential ideals K of A. Such algebras first arose in the study of derivations and were formally introduced by Pedersen in [17], where he proves that every derivation on a separable C*-algebra A extends to an inner derivation of $M_{\rm loc}(A)$. The question of whether every derivation of $M_{\rm loc}(A)$ is inner remains open for arbitrary separable C*-algebras.

A systematic study of local multiplier algebras is presented in the recent monograph by Ara and Mathieu [2]. One of the most important general facts concerning local multiplier algebras is that the centre $\mathcal{Z}\left(M_{\rm loc}(A)\right)$ of $M_{\rm loc}(A)$ is an AW*-algebra [1]. Although $M_{\rm loc}(A)$ itself need not be an AW*-algebra, Frank and Paulsen [8] have showed recently that $M_{\rm loc}(A)$ can nevertheless be realized as a C*-subalgebra of a certain minimal injective AW*-algebra: namely, the injective envelope $I(A)$ of A [9]. Further, even though $M_{\rm loc}(A)$ is not in general an AW*-algebra, there are examples in which $M_{\rm loc}(A)$ is actually a W*-algebra. We show herein that for separable C*-algebras, $M_{\rm loc}(A)$ is a W*-algebra if and only if A has a minimal essential ideal that is isomorphic to a C*-algebraic direct sum of elementary C*-algebras. This result also leads to a new proof of a theorem arising from work of Wright [20] and Hamana [12] that characterizes those separable A for which $I(A)$ is a W*-algebra.

As usual, we will denote by $B(H)$ and $K(H)$ the set of bounded and compact operators on a Hilbert space H.

The notion of injective envelope [9, 10, 16] first arose in two seminal papers of Arveson [4, 5]. One of the principal results of [5], the so-called boundary theorem, states that if E is an operator system acting on a Hilbert space H such that $K(H) \subset C^*(E)$, then the identity map on E has a unique completely positive extension to the algebra $C^*(E) \subset B(H)$ if and only if the quotient homomorphism onto the Calkin algebra is not completely isometric on E. This theorem is revisited in the present paper for a class of operator systems that generate discrete type I von Neumann algebras.

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Let $\mathcal{E}(A)$ denote the set of (closed) essential ideals of a C*-algebra A. For every $K \in \mathcal{E}(A)$, let $M(K)$ denote the multiplier algebra of K. If $K_1, K_2 \in \mathcal{E}(A)$ are such that $K_1 \subseteq K_2$, then $M(K_1) \supseteq M(K_2)$; thus, the family $\mathcal{E}(A)$ of essential ideals of A determines a downward-directed system of C*-algebras. The local multiplier algebra $M_{\rm loc}(A)$ of A is C*-algebraic direct limit that arises from $\mathcal{E}(A)$:

$$M_{\rm loc}(A) = \lim_{\rightarrow}\{M(K) \,:\, K \in \mathcal{E}(A)\}\,.$$

Every C*-algebra A is a C*-subalgebra of its injective envelope $I(A)$ [9]. Moreover, by [8, Corollary 4.3],

$$M_{\rm loc}(A) \;=\; \left(\bigcup_{K\in\mathcal{E}(A)} \{x \in I(A) \,:\, xK + Kx \subseteq K\} \right)^{-} \,,$$

where the closure is with respect to the norm topology of $I(A)$. Thus,

$$A \,\subseteq\, M_{\rm loc}(A) \,\subseteq\, I(A)$$

is an inclusion of C*-subalgebras. In [7], Frank showed an additional sequence of inclusions as C*-subalgebras:

$$A \,\subseteq\, M_{\rm loc}(A) \,\subseteq\, M_{\rm loc}\left(M_{\rm loc}(A)\right) \,\subseteq\, \overline{A} \,\subseteq\, I(A)\,.$$

In the inclusions above, $\overline{A}$ is the regular monotone completion [11] of A. For separable C*-algebras, $\overline{A}$ coincides with $\overline{A}^{\sigma}$, the regular monotone σ-completion [19] of A.

It is not known whether $\overline{A} \neq I(A)$ for separable C*-algebras A, but all other inclusions above can be proper. Most striking is the recent example of Ara and Mathieu [3] in which they show that $M_{\rm loc}(A) \neq M_{\rm loc}\left(M_{\rm loc}(A)\right)$ for a certain prime AF C*-algebra A.

Further relations are: $I(M_{\rm loc}(A)) = I(A)$ [8, Theorem 4.6] and $\mathcal{Z}(M_{\rm loc}(A)) = M_{\rm loc}(\mathcal{Z}(A)) = \mathcal{Z}(I(A))$ [7, Theorem 2] (since $\mathcal{Z}(M_{\rm loc}(A))$ is an AW*-algebra [2, Proposition 3.1.5] and, as it is abelian, is therefore injective).

1. $M_{\rm loc}(A)$ as a W*-algebra

It need not be true that $M_{\rm loc}(A)$ is an AW*-algebra. For example, $M_{\rm loc}(A) = A$ in the case where A is unital, simple, and separable—but AW*-algebras (of infinite dimension) are nonseparable. Although it is even less likely that $M_{\rm loc}(A)$ is a W*-algebra, this is precisely the case for a number of important examples (such as if A is a von Neumann algebra or if A can be represented as acting on a Hilbert space H in such a way as to contain every compact operator).

Theorem 1.2 below characterizes those separable C*-algebras that admit W*-algebra local multipliers. To prepare the way, the following lemma will be of use.

Lemma 1.1. *If A is a separable C*-algebra and if $M_{\rm loc}(A)$ is a W*-algebra, then $M_{\rm loc}(A)$ is of type I.*

Proof. Without loss of generality, we assume that $M_{\text{loc}}(A)$ is faithfully represented as a von Neumann algebra acting on a Hilbert space H. Thus,

$$A \subseteq A'' \subseteq M_{\text{loc}}(A) \subseteq \overline{A} \subseteq I(A),$$

as an inclusion of operator systems. Because A is order dense in $\overline{A}$ [11], A is also order dense in A''; that is,

$$h = \sup\{k \in A^+ \mid k \leq h\}, \quad \forall h \in A''.$$

Consequently, for any normal state ω on A'', $\omega(h) \geq \sup\{\omega(k) \mid k \in A^+, k \leq h\}$. Hence, any normal state ω on A'' that is faithful on A is also faithful on A''. This implies, by a theorem of Takesaki [18], that A'' is generated by its minimal projections and each minimal projection of A'' is contained in A. Hence, A'' is a discrete type I von Neumann algebra. Since type I AW*-algebras are injective, we conclude that $A'' = M_{\text{loc}}(A) = I(A)$. □

We shall employ the following notation from [2]. If $\{A_\alpha\}_{\alpha\in\Lambda}$is a family of C*-algebras, then

$$\prod_{\alpha\in\Lambda} A_\alpha = \{(a_\alpha)_\alpha : a_\alpha \in A_\alpha \text{ and } \sup_\alpha \|a_\alpha\| < \infty\};$$

$$\bigoplus_{\alpha\in\Lambda} A_\alpha = \{(a_\alpha)_\alpha : a_\alpha \in A_\alpha \text{ and } \forall\,\varepsilon > 0 \text{ only finitely many } a_\alpha \text{ satisfy } \|a_\alpha\| > \varepsilon\}.$$

Note that the direct product $\prod_\alpha A_\alpha$ and the direct sum $\bigoplus_\alpha A_\alpha$ are C*-algebras and $\bigoplus_\alpha A_\alpha$ is an ideal of $\prod_\alpha A_\alpha$.

The next theorem is one of the main results of this paper. Recall that an elementary C*-algebra is one that is isomorphic to $K(H)$ for some Hilbert space H.

Theorem 1.2. *The following statements are equivalent for a separable C*-algebra A.*

1. $I(A)$ *is a W*-algebra.*
2. $M_{\text{loc}}(A)$ *is a W*-algebra.*
3. $M_{\text{loc}}(A)$ *is a discrete type I W*-algebra.*
4. A *contains a minimal essential ideal that is isomorphic to a direct sum of elementary C*-algebras.*

Proof. (1) ⇒ (4). As A is a C*-algebra whose injective envelope $I(A)$ is a W*-algebra, $\overline{A}$ is also a W*-algebra (because a monotone closed C*-subalgebra of a von Neumann algebra is a von Neumann algebra [13]). Since A is separable, $\overline{A}$ has a countable order-dense subset (Wright notes in [21, page 84] that the equivalence of the separability and having a countable order-dense subset follows from Theorem 4.3 of [19]). Hence, by [21, Proposition A], the set of pure states of $\overline{A}$ (in the weak* topology) is hyperseparable. Since hyperseparability implies separability, another theorem of Wright [20, Corollary 7] shows that $\overline{A}$ is isomorphic to $\prod_n B(H_n)$ (a countable product), with each H_n separable. Further, since $\prod_n B(H_n)$ is injective, it follows that $I(A) = \overline{A} = \prod_n B(H_n)$. Finally, Lemma 3.1(iii) of [12] yields that $\bigoplus_n K(H_n) \subseteq A \subset \prod_n B(H_n)$. The minimality of $\bigoplus_n K(H_n)$ is given by [12, Proposition 3.3].

(4) $\Rightarrow$ (3). Suppose that A has a minimal essential ideal K such that $K \cong \bigoplus_n K(H_n)$. Therefore, by [2, Lemma 1.2.1],

$$M(K) \;=\; M\left(\bigoplus_n K(H_n)\right) \;=\; \prod_n M\left(K(H_n)\right) \;=\; \prod_n B(H_n)\,,$$

which shows that $M(K)$ is a (discrete type I) W*-algebra. Furthermore, because K is a minimal essential ideal of A, $M(K) = M_{\rm loc}(A)$ by [2, Remark 2.3.7]. Hence, $M_{\rm loc}(A)$ is a discrete type I W*-algebra.

The implication (3) $\Rightarrow$ (2) is trivial, and the implication (2) $\Rightarrow$ (3) is Lemma 1.1.

(3) $\Rightarrow$ (1). Type I AW*-algebras are injective, and so the inclusion $A \subseteq M_{\rm loc}(A) \subseteq I(A)$ (with $M_{\rm loc}(A)$ injective) implies that $M_{\rm loc}(A) = I(A)$, whence $I(A)$ is a W*-algebra. □

Corollary 1.3. *If any one of the equivalent conditions in Theorem 1.2 holds for a separable C*-algebra A, then*

$$M_{\rm loc}(A) \;=\; M_{\rm loc}\left(M_{\rm loc}(A)\right) \;=\; \overline{A} \;=\; I(A)\,.$$

Proof. Assume that any one of the statements (1)–(4) in Theorem 1.2 holds. Then $M_{\rm loc}(A)$ is an injective W*-algebra. However, $A \subseteq M_{\rm loc}(A) \subseteq I(A)$ as C*-subalgebras, and so by definition of the injective envelope, it must be that $M_{\rm loc}(A) = I(A)$, which proves that $M_{\rm loc}(A) = M_{\rm loc}\left(M_{\rm loc}(A)\right) = \overline{A} = I(A)$. □

Corollary 1.4. *If A is a separable, prime C*-algebra, then exactly one of the following two statements holds:*

(1) *$I(A) \cong B(H)$, for some separable Hilbert space H;*
(2) *$I(A)$ is a wild type III AW*-factor.*

In particular, if A has no nonzero postliminal ideals, then $I(A)$ is a wild type III AW*-factor.

Proof. Because A is prime, $I(A)$ is an AW*-factor [11, Theorem 7.1]. This factor cannot be of type II for the following reasons.

If $I(A)$ is a finite type II AW*-factor, then the identity $1 \in I(A)$ is a finite projection, and so 1 is a finite projection in $\overline{A}$ as well. Therefore, $\overline{A}$ is of type I [15, Theorem 2]. But type I algebras are injective; hence $\overline{A} = I(A)$, contradicting that $I(A)$ is of type II. Thus, assume that $I(A)$ is a type II$_\infty$ AW*-factor. Since $I(A)$ admits a faithful state (because A is separable), $I(A)$ is a W*-algebra by [6]. So Theorem 1.2 implies that $I(A)$ is of type I, which is a contradiction. Hence, $I(A)$ is a factor of either type I or type III.

In the case where $I(A)$ is of type I we have $I(A) \cong B(H)$ for some Hilbert space H, because all type I AW*-factors have this form [14, Theorem 2]. Indeed, in this case, $\overline{A} = I(A) \cong B(H)$; since $\overline{A}$ is countably decomposable, H can be chosen to be separable.

If $I(A)$ is not of type I, then the type III AW*-factor $I(A)$ cannot be a W*-algebra, by Theorem 1.2. Every AW*-factor that is not W*-algebra is wild [21]; hence, $I(A)$ is wild.

Finally, if A is prime and has a nonzero postliminal ideal, then $I(A)$ is of type I [12]. Thus, a prime separable C*-algebra with no nonzero postliminal ideals must have a wild type III injective envelope. □

2. A Version of the Boundary Theorem

If E is an operator system, then the C*-envelope [10, 16] of E is the C*-subalgebra $C^*_{\rm env}(E)$ of $I(E)$ generated by E. The C*-algebra $C^*_{\rm env}(E)$ is independent of the choice embedding of E into an injective envelope (I,κ) of E; thus, the notation $C^*_{\rm env}(E)$ is unambiguous.

The aim of the present section is to prove the following result.

Theorem 2.1. *Let $E \subseteq B(H)$ be an operator system for which the von Neumann algebra E'' is generated by its minimal projections, each of which is contained in the C*-subalgebra $C^*(E)$ of $B(H)$ generated by E. Then $I(E)$ is a type I W*-algebra and*

$$I(E) \cong E'' \quad \text{and} \quad C^*_{\rm env}(E) \cong C^*(E)\,.$$

Before turning to the proof of Theorem 2.1, recall that the original motivation for the concept of injectivity is Arveson's Hahn–Banach Extension Theorem [4] for completely positive linear maps, and that the idea of an injective envelope stems from Arveson's theory of boundary representations [5]. In Arveson's work on boundary representations, the operator systems were often realized as irreducible operator systems in $B(H)$ and their generated C*-algebras $C^*(E)$ were sometimes assumed to have nontrivial intersection with—and hence contain—the ideal $K(H)$ of compact operators. In this spirit, Theorem 2.1 is a generalization of the boundary theorem from $B(H)$ to discrete type I von Neumann algebras.

Two preliminary results are needed for the proof of Theorem 2.1. The first result is a proposition of Hamana that is a useful criterion for determining when an injective operator system I containing E is an injective envelope.

Proposition 2.2. ([9, Lemma 3.7]) *Consider an inclusion $E \subseteq I$ of operator systems, where I is injective. Then the following statements are equivalent.*

1. *I is an injective envelope of E.*
2. *The only completely positive linear map $\omega : I \to I$ for which $\omega|_E = id_E$ is the identity map $\omega = id_I$.*

The second preliminary result is a kind of partial converse to the main result of [18].

Lemma 2.3. *Suppose that A is a C*-subalgebra of a von Neumann algebra M and that $M = A''$. If M is generated by its minimal projections, each of which is contained in A, then A is order dense in M.*

Proof. Choose a nonzero $h \in M^+$ and consider the set

$$\mathcal{F} = \{\, (k_i) \subset A^+ : \sum_{\text{finite}} k_i \le h\}.$$

There is a strictly positive λ in the spectrum $\sigma(h)$ of h. Let $e \in M$ be the spectral projection $e = e^h([\lambda, \infty))$, where e^h denotes the spectral resolution of h. Thus, $0 \neq \lambda e \leq he$. Moreover, e majorizes a minimal projection p of M; by hypothesis, $p \in A$. Thus, $0 \neq \lambda p = e(\lambda p)e \leq e(\lambda)e = \lambda e \leq he \leq h$, and so $\lambda p \in \mathcal{F}$, which proves that $\mathcal{F} \neq \emptyset$. It is clear that $\mathcal{F}$ is inductive under inclusions of those families and so, by Zorn's Lemma, $\mathcal{F}$ has a maximal family W. Since every finite sum of this family is less than h,

$$y = \sup \left\{ \sum_{k \in K} k : K \text{ is finite and } K \subset W \right\} \leq h.$$

If $y \neq h$, then $h - y \in M^+$, and so by the first paragraph there exists nonzero $k \in A^+$ such that $k \leq h - y$. If it were true that $k \in W$, then for each net (h_i) of those finite sums of elements in W such that $h_i \nearrow y$, the net $(h_i + k) \nearrow y + k$, which contradicts the fact that y is the supremum. Hence, $k \notin W$. But if $k \notin W$, then the family W is not maximal, which is again a contradiction. Therefore, it must be that $y = h$, which proves that A is order dense in M.

Theorem 2.4. *If $A \subseteq B(H)$ is a C^*-algebra and if $M = A''$ is generated by its minimal projections, each of which is contained in A, then $\varphi = \mathrm{id}_M$ for every completely positive linear map $\varphi : M \to M$ for which $\varphi_{|A} = \mathrm{id}_A$.*

Proof. Observe that because $\varphi : M \to M$ is a unital completely positive map that preserves A, φ has the following property:

$$\varphi(xk) = \varphi(x)k, \text{ for every } k \in A.$$

This fact follows from the Cauchy-Schwarz inequality and from the fact that A is in the multiplicative domain of φ [16, Theorem 3.18]. Using this fact we shall deduce below that

$$x \geq 0 \text{ if and only if } \varphi(x) \geq 0. \tag{2.1}$$

Indeed, one implication is obvious from the positivity of φ. To prove the other implication, assume that $\varphi(x) \geq 0$. Thus, $\mathrm{Im}\,(\varphi(x)) = \varphi(\mathrm{Im}\,(x)) = 0$. Let $z = \mathrm{Im}\,(x)$ and write $z = z^+ - z^-$, where $z^+, z^- \in M^+$ are such that $z^+z^- = z^-z^+ = 0$.

Our first goal is to prove that $z^+ = 0$. Suppose, on the contrary, that $z^+ \neq 0$. Thus, there is a strictly positive λ in the spectrum of z^+; hence, there is a spectral projection $p \in M$ such that $0 \neq \lambda p \leq pz^+ = z^+p$. Note that $z^-p = 0$, as the projection p is in the von Neumann algebra generated by z^+ and $z^+z^- = z^-z^+ = 0$. Let $q \in A$ be an arbitrary minimal projection of M and consider the projection $p \wedge q \in M$. Because $p \wedge q \leq q$ and q is minimal, either $p \wedge q = 0$ or $p \wedge q = q$. We will show that the latter case cannot occur (under the conventional assumption that minimal projections are defined to be nonzero). Assume that it is true that $p \wedge q = q$. Then $0 \neq q = p \wedge q \leq p$. Pre- and post-multiply the inequality $\lambda q \leq \lambda p \leq z^+p = zp$ by q to obtain $\lambda q \leq q(zp)q \leq qzq$. Note that $\varphi(zq) = \varphi(z)q$ (because A is in the multiplicative domain of φ) and that $\varphi(z) = 0$ (by hypothesis). Likewise, for any hermitian $y \in M$, $\varphi(qy) = \varphi(yq)^* = q\varphi(y)$. Thus, $\varphi(qzq) = q\varphi(z)q = 0$ and $0 \leq \lambda q = \varphi(\lambda q) \leq q\varphi(z)q = 0$. This implies that $q = 0$, which contradicts the fact that q is minimal and, thus, nonzero. Therefore, it must be that $p \wedge q = 0$, for every minimal projection q of M. Because every nonzero projection in M majorizes a minimal projection, we conclude that $p = 0$, in contradiction to the fact that p is a nonzero spectral projection of z^+. Hence, it must be that $z^+ = 0$.

A similar argument shows that $z^- = 0$. We can find a nonzero $\lambda \in \mathbb{R}^+$ and a minimal projection $q \in A$ such that $qzq \leq -\lambda q$; thus $-\lambda q = \varphi(-\lambda q) \geq \varphi(qzq) = q\varphi(z)q = 0$, and again $q = 0$.

We conclude that $z = 0$, which implies that x is selfadjoint. It remains to show that x is positive. Assume that x is not positive. Thus, there exists a nonzero spectral projection in the negative part of $\sigma(x)$; by taking once again a suitable minimal subprojection q, we can find $\lambda > 0$ such that $qxq \leq -\lambda q$. But then $\varphi(qxq) \leq -\lambda q$; and on the other hand, $\varphi(qxq) = q\varphi(x)q \geq 0$. The contradiction implies that no such q can exist, and so $x \geq 0$.

From (2.1) and the fact the φ preserves A, we have that for $k \in A$, $k \leq x$ if and only if $k \leq \varphi(x)$. Lemma 2.3 asserts that A is order dense in M. Hence, $\varphi(x) = x$ for every $x \in M^+$, which implies that φ is the identity map on M. □

Proof of Theorem 2.1. By hypothesis, $C^*(E)$ contains all the minimal projections that generate E''. Theorem 2.4 together with Proposition 2.2 show that E'' is an injective envelope for E. Further, there is a completely positive projection ϕ on $B(H)$ with range E''. Hence, if $x, y \in E''$, then $x \circ y$—the product of x and y in the C*-algebra $I(E)$—is given by $x \circ y = \phi(xy) = xy$, since E'' is an algebra. Thus, $E'' = I(E)$ and $C^*(E)$ is precisely $C^*_{\text{env}}(E)$. □

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