Approximate amenability of Segal algebras

Mahmood Alaghmandan

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Approximate amenability

**Ghahramani and Loy [2004]:**
A Banach algebra $\mathcal{A}$ is said to be *approximately amenable* if for every $\mathcal{A}$-bimodule $X$ and every bounded derivation $D : \mathcal{A} \to X$, there exists a net $(D_\alpha)$ of inner derivations such that

$$\lim_{\alpha} D_\alpha(a) = D(a) \quad \text{for all } a \in \mathcal{A}.$$
Abstract Segal algebras

We say \((\mathcal{B}, \| \cdot \|_{\mathcal{B}})\) to be an *abstract Segal algebra* of Banach algebra \((\mathcal{A}, \| \cdot \|_{\mathcal{A}})\) if

1. \(\mathcal{B}\) is a dense left ideal in \(\mathcal{A}\).
2. There exists \(M > 0\) such that \(\|b\|_{\mathcal{A}} \leq M\|b\|_{\mathcal{B}}\) for each \(b \in \mathcal{B}\).
3. There exists \(C > 0\) such that \(\|ab\|_{\mathcal{B}} \leq C\|a\|_{\mathcal{A}}\|b\|_{\mathcal{B}}\) for all \(a \in \mathcal{A}\), \(b \in \mathcal{B}\).

\(\mathcal{B}\) is a proper subalgebra of \(\mathcal{A}\), we call it a *proper* abstract Segal algebra of \(\mathcal{A}\).
Segal algebras on locally compact groups

Let $G$ be a locally compact group. A linear subspace $S^1(G)$ of $L^1(G)$, the group algebra of $G$, is said to be a Segal algebra on $G$, if it satisfies the following conditions:

1. $S^1(G)$ is dense in $L^1(G)$.
2. $S^1(G)$ is a Banach space under some norm $\| \cdot \|_{S^1}$ and $\| f \|_{S^1} \geq \| f \|_1$ for all $f \in S^1(G)$.
3. $S^1(G)$ is left translation invariant and the map $x \mapsto L_x f$ of $G$ into $S^1(G)$ is continuous when $L_x f(y) = f(x^{-1}y)$.
4. $\| L_x f \|_{S^1} = \| f \|_{S^1}$ for all $f \in S^1(G)$ and $x \in G$.

Note that every Segal algebra is an abstract Segal algebra of $L^1(G)$ with convolution product.

Similarly, we call a Segal algebra on $G$ to be proper if it is a proper subalgebra of $L^1(G)$. 
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The history of approximate amenability of Segal algebras

Dales, Loy, and Zhang [2006] and Dales and Loy [2010]:
Certain Segal algebras on $\mathbb{T}$ and $\mathbb{R}$ are not approximately amenable.

Conjectured:
No proper Segal algebra on $\mathbb{T}$ is approximately amenable.

Choi and Ghahramani [2011]:
No proper Segal algebra on $\mathbb{T}^d$ or $\mathbb{R}^d$ is approximately amenable.
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A nice criterion

Choi and Ghahramani [2011]:

The criterion

Let $\mathcal{B}$ be a proper abstract Segal algebra of $\mathcal{A}$. If there exists a sequence $(u_n)_{n \geq 1} \subseteq \mathcal{B}$ such that:

- $u_n u_{n+1} = u_n = u_{n+1} u_n$.
- $\sup_n \|u_n\|_{\mathcal{A}} < \infty$.
- $\sup_n \|u_n\|_{\mathcal{B}} = \infty$.

Then $\mathcal{B}$ is not approximately amenable.
1. Abelian groups

2. Compact groups
A(G) = \{ f \ast \check{g} : f, g \in L^2(G) \}.

Applying Fourier transform

\[ \mathcal{F} : L^1(G) \to A(\hat{G}) \]

we may transform each proper Segal algebra to a proper abstract Segal algebra \( SA(\hat{G}) \) of \( A(\hat{G}) \).

So we study the abstract Segal algebras of Fourier algebra!
Segal algebras of locally compact abelian groups

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**Theorem**

Let $G$ be a locally compact abelian group. Then every proper Segal algebra of $G$ is not approximately amenable.

**Proof.** Let $S^1(G)$ a proper Segal algebra of $G$, then

$$SA(\hat{G}) = \mathcal{F}(S^1(G)).$$

We use that criterion to show that $SA(\hat{G})$ is not approximately amenable.
De la vallée poussin kernel

We are looking for \((u_n)_n \subseteq SA(\hat{G})\) such that:

- \(u_n u_{n+1} = u_n = u_{n+1} u_n\).
- \(\sup_n \| u_n \|_{A(\hat{G})} < \infty\).
- \(\sup_n \| u_n \|_{SA(\hat{G})} = \infty\).

Generating \(u_n\)'s by \(A(\hat{G})\) properties.

Trivially \(\implies u_n u_{n+1} = u_n = u_{n+1} u_n\).

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Leptin condition

$G$ is amenable if and only if it satisfies \textit{Leptin condition} i.e. for every $\epsilon > 0$ and compact set $K \subseteq G$, there exists a relatively compact neighborhood $V$ of $e$ such that $\lambda(KV)/\lambda(V) < 1 + \epsilon$.

\textbf{Leptin condition on abelian group} $\hat{G} \iff \sup_n \|u_n\|_{A(\hat{G})} < \infty$.

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Let \( G \) be a locally compact abelian group. Then every proper Segal algebra of \( G \) is not approximately amenable.
1 Abelian groups

2 Compact groups
Compact groups

Let $G$ be a compact group, $\hat{G} = \{\pi\}$ the set of all irreducible unitary representations of $G$. $\pi : G \to B(\mathcal{H}_\pi)$ $(\mathcal{H}_\pi = \mathbb{C}^{d_\pi})$. There is indeed kind of convolution on $\hat{G}$ makes it a hypergroup.

Define measure $h$ on discrete set $\hat{G}$, where $h(\pi) = d_\pi$. Indeed $L^1(\hat{G}, h)$ forms a Banach algebra with that convolution.

For each $\pi \in \hat{G}$, we integrate $\pi : L^1(G) \to B(\mathcal{H}_\pi)$, where $f \mapsto \hat{f}(\pi)$.

$$\mathcal{F} : f \in L^1(G) \mapsto \begin{bmatrix} \hat{f}(\pi_1) & 0 & \cdots & 0 & \cdots \\ 0 & \hat{f}(\pi_2) & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & 0 & \cdots \\ 0 & 0 & \cdots & \hat{f}(\pi_n) & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{bmatrix}$$
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\end{bmatrix}\]
Center of the group algebra

\[ ZL^1(G) := \{ f \in L^1(G) : f(yxy^{-1}) = f(x) \text{ for all } x \in G \}. \]

\[ \mathcal{F} : f \in ZL^2(G) \subseteq ZL^1(G) \mapsto \begin{bmatrix} \alpha_{\pi_1} l_{\pi_1} & 0 & \cdots & 0 & \cdots \\ 0 & \alpha_{\pi_2} l_{\pi_2} & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & 0 & \cdots \\ 0 & 0 & \cdots & \alpha_{\pi_n} l_{\pi_n} & \cdots \\ \vdots & \vdots & \cdots & \vdots & \ddots \end{bmatrix} \]

Note:

\[ \sum_{\pi \in \hat{G}} \alpha_{\pi}^2 d_{\pi} < \infty \implies (\alpha_{\pi})_{\pi \in \hat{G}} \in L^2(\hat{G}, h). \]
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Note:

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Fourier algebra on $\hat{G}$

Using $\mathcal{F}$, we define $T : ZL^2(G) \to L^2(\hat{G}, h)$, where $T(f) = (\alpha_\pi)_\pi (h(\pi) = d_\pi)$.

So we can extend this map from $ZL^1(G)$ onto $L^2(\hat{G}, h) \ast L^2(\hat{G}, h)$. We see that

$$A(\hat{G}) := \{ f \ast \check{g} : f, g \in L^2(\hat{G}, h) \}$$

is a Banach algebra with pointwise product on $\hat{G}$.

Fourier space on hypergroups defined by Muruganandam [2007].

Good Things!

$T(ZL^1(G)) = A(\hat{G})$ as Banach algebras.
Fourier algebra on $\hat{G}$

Using $\mathcal{F}$, we define $\mathcal{T} : ZL^2(G) \to L^2(\hat{G}, h)$, where $\mathcal{T}(f) = (\alpha_\pi)_\pi$ ($h(\pi) = d_\pi$).

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**Good Things!**

$\mathcal{T}(ZL^1(G)) = A(\hat{G})$ as Banach algebras.
Let $G$ be a compact group. Let $S^1(G)$ a proper Segal algebra of $G$, then

$$\mathcal{T}(ZS^1(G)) \subseteq A(\hat{G}).$$

Let us apply the earlier criterion on $\mathcal{T}(ZS^1(G))$.
We can generate a sequence \((u_n) \subseteq \mathcal{T}(ZS^1(G))\) such that

1. \(u_n u_{n+1} = u_n\).
2. \(\sup_n \|\mathcal{T}^{-1}(u_n)\|_{S^1(G)} = \infty\).

To satisfy the last condition, we need Leptin condition for hyper group \(\hat{G}\). We define *Leptin condition* for hypergroups and study that for some special hyper groups:

**Leptin condition for hypergroups**

We say that \(H\) satisfies *Leptin condition* if for every compact subset \(K\) of \(H\) and \(\epsilon > 0\), there exists a measurable set \(V\) in \(H\) such that

\[
0 < h(V) < \infty \text{ and } h(K \ast V)/h(V) < 1 + \epsilon.
\]

If \(\hat{G}\) satisfies Leptin condition \(\implies\)

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We can generate a sequence \((u_n)_n \subseteq T(ZS^1(G))\) such that

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If \(\hat{G}\) satisfies Leptin condition \(\implies\)
- \(\sup_n \|\mathcal{T}^{-1}u_n\|_{L^1(G)} < \infty\).
Theorem

Let $G$ be a compact group such that $\hat{G}$ satisfies Leptin condition. Then every proper Segal algebra on $G$ is not approximately amenable.
For which compact groups $G$, $\hat{G}$ satisfies Leptin condition?

**$SU(2)$**

Leptin condition is held for $SU(2)$.

**Product of compact groups**

Let $\{G_i\}_{i \in I}$ be a family of compact groups whose duals have Leptin condition and $G = \prod_{i \in I} G_i$ is their product equipped with product topology. Then $\hat{G}$ satisfies Leptin condition.

**Example.** If $G$ is the product of a family of finite groups, $\hat{G}$ satisfies Leptin condition.


Thank You!